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A SOLVABLE QSDE THROUGH SEMIGROUPS OF OPERATORS AND SOME PHYSICAL APPLICATIONS

O. González-Gaxiola[1 §], José A. Santiago[2], G. Chacón-Acosta[3]
[1,2,3]Department of Applied Mathematics and Systems
UAM–Cuajimalpa
Artificios 40, México, D.F. 01120, MEXICO

Abstract: In this paper we give a characterization, through semigroups theory, of the solution of a quantum stochastic differential equation (QSDE). In the physical interpretation of the problem, we show that the group that characterizes the quantum dynamics appears as the strong limit a family of translations perturbed by a Gaussian potential. Finally, we use the model to study a two-level atom in an electromagnetic field.

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1. Introduction

The QSDE, like the Hudson-Parthasarathy equation, has been mathematically studied from several points of view [1], [2], [7], [4], [8], but has not been solved in an exactly manner. In this work, we formalize mathematically the idea of solution given in [1] , we also give a physical interpretation in the case $L = 0$ and obtain the quantum dynamics of a two-level atom interacting with an electromagnetic field [5]. This will be done through operators semigroup

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§Correspondence author

theory, characterizing the infinitesimal operator that generates the dynamics.

2. Preliminaries

In this section we give the definition of the Fock space, which is a new Hilbert space builded from a given Hilbert space.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces; for each $\varphi_1 \in \mathcal{H}_1$ and $\varphi_2 \in \mathcal{H}_2$, the product $\varphi_1 \otimes \varphi_2$ denotes the bilinear form that acts on $\mathcal{H}_1 \times \mathcal{H}_2$ by the following rule:

$$(\varphi_1 \otimes \varphi_2)(\psi_1, \psi_2) = \langle \psi_1, \varphi_1 \rangle_1 \langle \psi_2, \varphi_2 \rangle_2,$$

where $\langle \cdot, \cdot \rangle_i$ is the inner product in the Hilbert space $\mathcal{H}_i$.

Let $\mathcal{E}$ be the set of all finite linear combinations:

$$\mathcal{E} = \{\sum_{i=1}^{n} \alpha_i (\varphi_{i_1} \otimes \varphi_{i_2}) : \ \forall i = 1, ..., n \quad \varphi_{i_1} \in \mathcal{H}_1, \ \varphi_{i_2} \in \mathcal{H}_2, \ \alpha_i \in \mathbb{C}\}.$$

We define an inner product $\langle \cdot, \cdot \rangle$ over $\mathcal{E}$, as follows:

$$\langle \varphi \otimes \psi, \eta \otimes \mu \rangle = \langle \varphi, \eta \rangle_1 \langle \psi, \mu \rangle_2.$$

This definition extends to all $\mathcal{E}$ by the bilinearity property and it can be proved that $\langle \cdot, \cdot \rangle$ is positive definite and well defined.

Definition 1. Let be the space $\mathcal{H}_1 \otimes \mathcal{H}_2$ as the completation of $\mathcal{E}$ under the inner product $\langle \cdot, \cdot \rangle$ defined above. $\mathcal{H}_1 \otimes \mathcal{H}_2$ will be called the tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$; $\mathcal{H}_1 \otimes \mathcal{H}_2$ is a Hilbert space.

We have the following two propositions whose proofs can be found in [6].

Proposition 1. *If $\{\varphi_k\}$ and $\{\psi_l\}$ are orthonormal basis of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, $\{\varphi_k \otimes \psi_l\}$ is an orthonormal basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$.*

Proposition 2. *Let (M_1, μ_1) and (M_2, μ_2) be measure spaces, and let consider the spaces $L^2(M_1, d\mu_1)$ and $L^2(M_2, d\mu_2)$. Then, there exists a unique isomorphism from $L^2(M_1, d\mu_1) \otimes L^2(M_2, d\mu_2)$ to $L^2(M_1 \times M_2, d\mu_1 \otimes d\mu_2)$ such that, $f \otimes g \mapsto f \cdot g$.*

Definition 2. If $\{\mathcal{H}_n\}_{n=1}^{\infty}$ is a sequence of Hilbert spaces, let us define the direct sum space as the set

$$\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n = \left\{ \{x_n\}_{n=1}^{\infty} : x_n \in \mathcal{H}_n, \quad \sum_{n=1}^{\infty} \|x_n\|_{\mathcal{H}_n}^2 < \infty \right\}.$$

Again, the set $\mathcal{H}$ is a Hilbert space under the inner product $\langle\cdot,\cdot\rangle : \mathcal{H}\times\mathcal{H} \longrightarrow \mathbb{C}$ given by

$$\langle \{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}\rangle_{\mathcal{H}} = \sum_{n=1}^{\infty} \langle x_n, y_n\rangle_{\mathcal{H}_n}.$$

Definition 3. Let $\mathcal{H}$ be a Hilbert space, we denote by $\mathcal{H}^{\otimes^n}$ the tensorial product of n-th order, i.e, $\mathcal{H}^{\otimes^n} = \mathcal{H}\otimes\mathcal{H}\otimes\cdots\otimes\mathcal{H}$. Let $\mathcal{H}^0 = \mathbb{C}$, and we introduce

$$\Gamma(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes^n}.$$

$\Gamma(\mathcal{H})$ is called the Fock space over $\mathcal{H}$; this space will be separable if $\mathcal{H}$ is separable.

If $\mathcal{H} = L^2(\mathbb{R})$, then an element $\psi \in \Gamma(L^2(\mathbb{R}))$ is a sequence of functions that is of the form

$$\psi = \{\psi_0, \psi_1, \psi_2, \cdots\}$$

where $\psi_0 \in \mathbb{C}$, and for all k, $\psi_k \in L^2(\mathbb{R}^k)$. From Definition 2 we have that the inner product and the norm in $\Gamma(\mathcal{H})$ are:

$$\begin{aligned}
\langle \psi, \varphi\rangle_{\Gamma(\mathcal{H})} =& \overline{\psi_0}\varphi_0 + \sum_{k=1}^{\infty} \langle \psi_k, \varphi_k\rangle_{L^2(\mathbb{R}^k)}, \\
\|\psi\|^2_{\Gamma(\mathcal{H})} =& |\psi_0|^2 + \sum_{k=1}^{\infty} \langle \psi_k, \psi_k\rangle_{\mathcal{H}^{\otimes k}} \\
=& |\psi_0|^2 + \sum_{k=1}^{\infty} \int_{\mathbb{R}^k} |\psi_k(x_1, x_2, \cdots, x_k)|^2 dx_1 \cdots dx_k < \infty.
\end{aligned}$$

To conclude this section we define for each $g \in \mathcal{H}$ the exponential vector $\psi(g)$, of argument g, whose components are given by:

$$\psi_0 = 1, \quad \psi_k(g) = \frac{1}{\sqrt{k!}} g\otimes\cdots\otimes g;$$

it can be shown that the set of exponential vectors over the Hilbert space $\mathcal{H}$ is dense on $\Gamma(\mathcal{H})$.

3. Study of the QSDE

Let us consider the following Quantum Stochastic Differential Equation (QSDE) [4], [8]:

$$\begin{cases} dV_t = [(W-I)d\Lambda_t - L^*WdA_t + LdA_t^\dagger - (iH + \frac{1}{2}L^*L)dt]V_t, \\ V_0 = I. \end{cases} \tag{3.1}$$

where W, L and H are the operators in the separable Hilbert space $\mathcal{H}$, while $d\Lambda_t$, dA_t, $dA_t^\dagger$ are the basic stochastic differentials in a symmetric Fock space.

According to the main results of quantum stochastic calculus [8], the solution of the previous QSDE, $\{V_t\}_{t\geq 0}$, is an adapted process of operators in $\Gamma^s(L^2(\mathbb{R}_+)) \otimes \mathcal{H}$, that describes the total evolution of the quantum system along with its environment.

There is a one-parametric group $\{U_t\}_{t\in\mathbb{R}}$, canonically associated with$\{V_t\}_{t\geq 0}$; [2].

In order to construct U_t is necessary to consider the second quantization Θ_t, over the symmetric Fock space

$$\Gamma\left(L^2(\mathbb{R})\right) = \Gamma\left(L^2(\mathbb{R}_-)\right) \otimes \Gamma\left(L^2(\mathbb{R}_+)\right),$$

of the unitary strongly continuous group of the translation operator θ_t on $L^2(\mathbb{R})$; which is defined for each $t \in \mathbb{R}$ by

$$\theta_t v(r) = v(r+t), \text{ for all } v \in L^2(\mathbb{R}) \text{ continuous},$$

$$\Theta_t \psi(v) = \psi(\theta_t v), \text{ for each } v \in L^2(\mathbb{R}),$$

where $\psi(v)$ is the exponential vector associated with v.

The operator Θ_t, can be identified with its extension to all space $\Gamma\left(L^2(\mathbb{R})\right) \otimes \mathcal{H}$, and satisfies the cocycle property of V_t with respect to Θ_t:

$$V_{s+t} = \Theta_s^* V_t \Theta_s V_s, \ \forall s, t \geq 0.$$

The strong continuity of V_t and Θ_t and the cocycle property of V_t, ensure the strong continuity of U_t.

We can define the unitary strong continuous one-parametric group $\{U_t\}_{t\in\mathbb{R}}$, by means of

$$U_t = \begin{cases} \Theta_t V_t & t \geq 0 \\ V_{|t|}^* \Theta_t & t < 0. \end{cases}$$

By the Stone theorem [9], U_t and Θ_t are generated by some self adjoint operators (their Hamiltonians), i.e., for each $t \in \mathbb{R}$ we formally have,

$$U_t = e^{-itF}, \quad \Theta_t = e^{-itE}.$$

While the operator E is known, the problem of completely characterize the operator F is not easy since, in general F is a singular operator and its domain is described by using boundary conditions.

4. Infinitesimal generator of $\{U_t\}_{t\in\mathbb{R}}$

The presents results were studied mathematically in [7]. There, it was considered the $L = 0$ case, and W was an unitary operator that commutes with the self adjoint and bounded operator H, that is the Hamiltonian of the system. The associated unitary group of the QSDE is defined for each $t \in \mathbb{R}$ by [1],

$$U_t\Psi = (\psi_0, U_{t,1}\psi_1, U_{t,2}\psi_2, ...)$$

where

$$\Psi = (\psi_0, \psi_1, \psi_2, ...) \in \Gamma^s\left(L^2(\mathbb{R}_*)\right) \otimes \mathcal{H}$$

with ψ_n continuous for each $n \geq 1$, $\psi_0 \in \mathbb{C}$ and $\mathbb{R}_* = \mathbb{R} \setminus \{0\}$. For each $n \geq 1$ we have,

$$U_{t,n} = \begin{cases} U_{t,n} & t \geq 0 \\ U^*_{|t|,n} & t < 0, \end{cases}$$

where

$$(U_{t,n}\psi_n)(x_1, ..., x_n) = e^{-itH} \bigotimes_{j=1}^{n} \{I_{(-t,0)}(x_j)v(x_j + t)W + I_{\mathbb{R}\setminus(-t,0)}(x_j)v(x_j + t)\} \otimes h,$$

where e^{-itH} and W can be identified with $I \otimes e^{-itH}$ and $I \otimes W$, respectively. I_A is the indicator function of the set A and $\psi_n(x_1, ..., x_n) = v(x_1) \otimes \cdots \otimes v(x_n) \otimes h$, $h \in \mathcal{H}$. The definition of $U_{t,n}$ can be extended by density to all $L^2(\mathbb{R}^n_*) \otimes \mathcal{H}$ for each $n \in \mathbb{N}$.

In [1], an idea was given of how to characterize the corresponding infinitesimal generator; however, there was not given a complete proof of such a result. Here we will outline a characterization of the generator of the unitary group U_t.

Let us consider the dense subspace of $\Gamma^s(L^2(\mathbb{R}_*)) \otimes \mathcal{H}$ given by

$$\mathbb{F} = \{\Psi \in \Gamma^s(L^2(\mathbb{R}_*)) \otimes \mathcal{H} : \exists k \in \mathbb{N} \text{ tal que } \psi_n = 0 \,\forall\, n > k\}$$

Let F_j be the operator $F_j = \sum_{k=1}^{j} i\partial_{x_k} + H$, defined as

$$D_j = \{\psi_j \in \mathcal{W}^s_{2,1}(\mathbb{R}^j_*) \otimes \mathcal{H} : \psi_j(0^-) = W\psi_j(0^+)\},$$

where, $\mathcal{W}^s_{1,2}$ is the set of vectors on $\Gamma^s\left(L^2(\mathbb{R}_*)\right)$ whose j-th component is element of the Sobolev space $\mathcal{W}_{2,1}(\mathbb{R}^j_*)$. $\mathcal{W}^s_{2,1}(\mathbb{R}^j_*)$ is the projection of $\mathcal{W}^s_{2,1}$ on j-particle space $L^2(\mathbb{R}^j_*)$.

Theorem 1. *The closure of the operator F on $\Gamma^s\left(L^2(\mathbb{R}_*)\right) \otimes \mathcal{H}$ given by $F\Psi = (0, F_1\psi_1, ..., F_n\psi_n, ...)$ with domain $\mathcal{D} = \{\Psi \in \mathcal{F} : \psi_j \in D_j, \quad j \in \mathbb{N}\}$, is the infinitesimal generator of the strongly continuous group $\{U_t\}_{t\in\mathbb{R}}$ in $\Gamma^s(L^2(\mathbb{R}_*)) \otimes \mathcal{H}$.*

In order to proof the previous theorem let us consider the one particle space, i.e., the first component of the Fock space.

Proposition 3. *The closure of the operator $F_1 = i\partial_x + H$ with domain $D_1 = \{\psi \in \mathcal{W}_{2,1}(\mathbb{R}_*) \otimes \mathcal{H} : \psi(0^-) = W\psi(0^+)\}$ is the infinitesimal generator of the strongly continuous group of unitary operators. $\{U_{t,1}\}_{t\in\mathbb{R}}$ in $L^2(\mathbb{R}_*)$.*

Proof. The proof is to show that D_1 is an essential domain for F_1. Because of a well known result for essential domains of infinitesimal generators [2]; is enough to proof the following:

1. D_1 is dense on $L^2(\mathbb{R}_*) \otimes \mathcal{H}$.
2. D_1 remains invariant under $U_{t,1}$.
3. $D_1 \subset D(F_1)$.

To illustrate the demonstration of these points, we will only consider the proof of point (3). Let $\psi \in D_1$, then

$$\begin{aligned} &\|\frac{1}{t}(U_{t,1} - I)\psi(x) - (i\partial_x + H)\psi(x)\|^2 \\ &\qquad = \int_{-\infty}^{-t} |\frac{1}{t}(U_{t,1} - I)\psi(x) - (i\partial_x + H)\psi(x)|^2 dx \\ &\qquad\quad + \int_{-t}^{0} |\frac{1}{t}(U_{t,1} - I)\psi(x) - (i\partial_x + H)\psi(x)|^2 dx \end{aligned}$$

$$+\int_0^\infty |\frac{1}{t}(U_{t,1}-I)\psi(x)-(i\partial_x+H)\psi(x)|^2dx.$$

The first and third integrals tends to zero as $t\to 0$, this is because of the fact that the operator iF_1 defined on $\mathcal{W}_{2,1}(\mathbb{R}_*)\otimes\mathcal{H}$ is the infinitesimal generator of the unitary group $(V_t\psi)(x)=e^{-itH}\psi(x+t)$, for each continuous function $\psi\in L^2(\mathbb{R}_*)\otimes\mathcal{H}$.

To perform the the second integral, we can introduce the function

$$\tilde{\psi}(x)=\begin{cases} W\psi(x) & ,x\geq 0\\ \psi(x) & ,x<0\end{cases}$$

by substituting in the second integral we obtain

$$\begin{aligned}&\int_{-t}^0 |\frac{1}{t}(U_{t,1}-I)\psi(x)-(i\partial_x+H)\psi(x)|^2dx\\ &\qquad=\int_{-t}^0 |\frac{1}{t}(e^{-itH}W\psi(x+t)-\psi(x))-(i\partial_x+H)\psi(x)|^2dx\\ &\qquad=\int_{-t}^0 |\frac{1}{t}(e^{-itH}\tilde{\psi}(x+t)-\tilde{\psi}(x))-(i\partial_x+H)\tilde{\psi}(x)|^2dx\to 0,\end{aligned}$$

when $t\to 0$, the last limit holds because the operator $-\partial_x+iH$ defined in $\mathcal{W}_{2,1}(\mathbb{R}_*)\otimes\mathcal{H}$ is the infinitesimal generator of the unitary group

$$(V_t\psi)(x)=e^{-itH}\psi(x+t),\quad \psi\in L^2(\mathbb{R}_*)\otimes\mathcal{H}.$$

In this way we have proved that $D_1\subset D(F_1)$. □

The proof of Theorem 1 can be completed by taking the second quantization of the unitary group and its generator in proposition 3. This method allows to determine the essential domain of the operator F.

5. Physical Applications

5.1. Linear Motion of a Photon with Local Interaction

If we consider the group $U_{t,1}$ in $L^2(\mathbb{R}_*)$, i.e., if $\mathcal{H}=\mathbb{C}$; and we take the operators $W=e^{i\theta}$and $H=0$ with θ a real nonzero, then we obtain the strongly continuous group of unitary operators

$$U^\theta_{t,1}=\begin{cases} U^{\theta+}_{t,1} & ,t\geq 0\\ U^{\theta*}_{|t|,1} & ,t<0\end{cases}$$

where, for each $t \geq 0$

$$\left(U_{t,1}^{\theta+}\psi\right)(x) = \psi(x+t)\{e^{i\theta}I_{(-t,0)}(x) + I_{\mathbb{R}\setminus(-t,0)}(x)\}.$$

Physically, this system models the evolution of a photon that moves across an optical axis (the real line), from right to left and, permanently interacts at the origin, with an external quantum system, which imprints to it a phase shift equal to $e^{i\theta}$.

According to proposition 3, the infinitesimal generator of $U_{t,1}^{\theta}$ is the closure of the operator $F_1^{\theta} = i\partial_x$ with domain $D_1 = \{\psi \in \mathcal{W}_{2,1}(\mathbb{R}_*) : \psi(0^-) = e^{i\theta}\psi(0^+)\}$.

If for each $t \in \mathbb{R}$ we define on $L^2(\mathbb{R}_*)$ the family of operators given by

$$U_{t,1}^{\theta,\alpha} = \begin{cases} U_{t,1}^{\theta,\alpha+} & ,t \geq 0 \\ U_{|t|,1}^{\theta,\alpha*} & ,t < 0 \end{cases}$$

where, for each $t \geq 0$

$$\left(U_{t,1}^{\theta,\alpha+}\psi\right)(x) = \psi(x+t)e^{i\theta\int_0^t V_\alpha(x+t-\tau)d\tau}$$

with $V_\alpha = (2\pi\alpha)^{-\frac{1}{2}}e^{-\frac{x^2}{2\alpha}}$, then the following propositions can be established:

Proposition 4. *The family $\{U_{t,1}^{\theta,\alpha}\}_{t\in\mathbb{R}}$ is a strongly continuous group of unitary operators in $L^2(\mathbb{R}_*)$ whose infinitesimal generator is the operator $F_1^{\theta,\alpha} = \partial_x + i\theta V_\alpha$ defined on $\mathcal{W}_{2,1}(\mathbb{R}_*)$.*

Proposition 5. *$U_{t,1}^{\theta,\alpha} \to U_{t,1}^{\theta}$ when $\alpha \to 0^+$ strongly in $\mathcal{B}\left(L^2(\mathbb{R}_*)\right)$. Then, $F_1^{\theta,\alpha} \to F_1^{\theta}$ when $\alpha \to 0^+$ strongly in the resolvers sense.*

According to these propositions, the dynamics generated by the operator F_1^{θ}, is the limit of the dynamics generated by the operator $F_1^{\theta,\alpha}$, when $\alpha \to 0^+$.

5.2. Two-Level Atom Model

In quantum optics the basic model consists of some physical system of interest, for example, a two-level atom, a cloud of atoms, or an atom in a cavity, in interaction with an electromagnetic field. The interaction between the electromagnetic field and the system is described by quantum electrodynamics [3].

Suppose we are studying a two-level atom in interaction with the electromagnetic field; in this case the operarots W, L y H acting over $\mathcal{H} = \mathbb{C}^2$, are given by

$$W = I, \quad L = \gamma\sigma_- = \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}, \quad H = \frac{\hbar\omega_0}{2}\sigma_z = \begin{pmatrix} \frac{\hbar\omega_0}{2} & 0 \\ 0 & -\frac{\hbar\omega_0}{2} \end{pmatrix},$$

where $\hbar$ is the Planck constant, $\gamma \geq 0$ is a decay parameter, ω_0 the atomic frecuency and $\hbar\omega_0$ the diference of energy between the two levels. The interaction with the electromagnetic field is through the QSDE:

$$\begin{cases} dV_t = [LdA_t^\dagger - L^* dA_t - (iH + \frac{1}{2}L^*L)dt]V_t \\ V_0 = I. \end{cases} \tag{5.1}$$

Now, taking into account the two-level atom model, we have

$$\begin{cases} dV_t = [\gamma\sigma_- dA_t^\dagger - \gamma\sigma_+ dA_t - \frac{\gamma^2}{2}\sigma_+\sigma_- dt - i\frac{\hbar}{2}\sigma_z]V_t \\ V_0 = I, \end{cases} \tag{5.2}$$

whose formal solution is:

$$V_t = \mathcal{T}exp\{\int_0^t \gamma\sigma_- dA_s^\dagger - \gamma\sigma_+ dA_s - i\frac{\hbar\omega_0}{2}\sigma_z ds\},$$

where $\mathcal{T}$ denotes the time-ordered product of operators.

6. Summary

By a different approach from that developed by A. Chebotarev [1] and [2], we studied the problem of characterizing the solution of a QSDE in the framework of the theory of operators. The method outlined in third section allows to characterize the infinitesimal generator of the unitary group $\{U_t\}_{t\in\mathbb{R}}$. This unitary group admits the physical interpretation described in the fourth section and it is the strong limit of a family of translations perturbed by a Gaussian potential. Formally, the family of infinitesimal generators of these unitary groups is the singular operator $\partial_x + \delta_0$ with δ_0 the Dirac delta measure concentrated at the origin; in our approach this singular operator was rigorously defined as the operator F_1^θ.

In physics, there also exist some systems that could be modeled through equations like (3.1), such is the case of the equation for the amplitude of a laser, which must consider noise effects due to both the quantum nature of the laser, and its interaction with the environment [10]. Such noise could be modeled through stochastic processes and modeled in a QSDE as (3.1); however, to complete the connection further studies are still required.

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