ELASTIC CURVES AND SURFACES UNDER LONG-RANGE FORCES: A GEOMETRIC APPROACH

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Using classical differential geometry, the problem of elastic curves and surfaces in the presence of long-range interactions }Φ, is posed. Starting from a variational principle, the balance of elastic forces and the corresponding projections $n_i \cdot \nabla \Phi$, are found. In the case of elastic surfaces, a force coupling the mean curvature with the external potential, $K\Phi$, appears; it is also present in the shape equation along the normal principal in the case of curves. The potential }Φ contributes to the effective tension of curves and surfaces and also to the orbital torque. The confinement of a curve on a surface is also addressed, in such a case, the potential contributes to the normal force through the terms $-\kappa \Phi - n \cdot \nabla \Phi$.

In general, the equation of motion becomes integro-differential that must be numerically solved.

Keywords: Elastic curves and surfaces; charged polymers.

1. Introduction

The elastic energy of a polymer chain, can be modeled as a geometric functional, invariant under reparametrizations, $\mathcal{H} = \int ds f(\kappa, \tau)$, $\kappa$ and $\tau$ being the Frenet-Serret curvatures and $s$ the arclength of the curve accounting the polymer. Working within the geometric framework of Frenet-Serret, an infinitesimal deformation of the curve can be decomposed in its perpendicular and tangential parts; the invariance under reparametrizations has the effect that just orthogonal projections play a role in the equilibrium shape equations while a tangential deformation yields only a boundary term. Is possible to integrate the Euler-Lagrange equations for the curvature in the case of plane curves. If the energy only depends on the curvature, the model is integrable, and then the torsion is a function only of curvature. Furthermore, Noether conserved charges of the model, associated with invariances of motions in space, can be obtained using this geometric formalism.

In certain processes in cell biology and in certain technological applications, the behavior of polymers under the influence of external fields is of great interest.
The problem of polymers chains constrained to confined geometries was already considered\textsuperscript{3}. However, there was not taken into account the role of a potential acting on the polymer. For example, in the case of DNA packaging, not only the entropic and elastic forces act on the polymer, but also the electrostatic force. Indeed, as the DNA is immersed in a solution with several other ions, the effective electrostatic potential can be modified as the Debye-Hückel potential\textsuperscript{4} $\Phi(r) \propto r^{-1}e^{-kr}$, that results from a linearized mean-field procedure. In general all polyelectrolytes, which are charged polymer chains, that are in solution have similar behavior. When the polymer chains are negatively charged, the charges produce their mutual repulsion and the polymer cannot be wrapped, so the chain is stretched\textsuperscript{5}. They also have certain behavior when the chain is close to a charged surface, this behavior has been used to model some biological interactions such as in nucleosome structures of DNA\textsuperscript{6}. The properties of the polyelectrolytes are well known, see e.g. refs.\textsuperscript{7–8}, however, the geometric properties under external long-range interactions are not yet studied in detail. In the present work we consider besides the elastic forces, the long-range interactions on curves and surfaces.

In section (2), starting from a variational principle, we obtain the Euler-Lagrange equations for elastic curves in presence of long range forces. As a result of translation and rotational invariance of the energy, the corresponding Noether charges are found. The example of elastic curves dependent on the curvature $\kappa$, and the particular case of planar curves are shown, for these cases a first integral of the shape equation can be found. In section (3), the analogous analysis for surfaces with long range forces is presented. The shape equation from a variational principle and the corresponding forces and torque are found. Explicit equations in the case of fluid membranes are presented. In section (4), we use the alternative method introducing auxiliary variables to regain the Euler-Lagrange equations of section (2). The equations of elastic curves in presence of long range forces, constrained to surfaces are presented. Moreover, the normal force that constraint the curve is found in terms of the geometric information. We conclude with some final remarks in section (5).

2. Curves

2.1. Shape equations

In the presence of long range interactions, an elastic curve can be modeled by the following energy functional

$$\mathcal{H} = \int ds f(\kappa, \tau) + \frac{1}{2} \int \int ds \, ds' h(r).$$

The first term $\mathcal{H}_a = \int ds f(\kappa, \tau)$, constitutes the elastic energy of a curve in $\mathbb{R}^3$, $x = X(s)$, parametrized by arclength $s$\textsuperscript{9}. The curvature $\kappa$ and torsion $\tau$ of the
curve are defined through the Frenet-Serret equations:
\[
\begin{align*}
\dot{t} &= \kappa n_1, \\
\dot{n}_1 &= -\kappa t + \tau n_2, \\
\dot{n}_2 &= -\tau n_1,
\end{align*}
\] (2)

here the dot indicates the derivative with respect to the arclength \(s\). The unit tangent
vector \(t = \dot{X}\), together with the unit normal vectors \(n_i\), define a local basis \(\{t, n_1, n_2\}\),
adapted to the curve. Among others, some relevant cases are the energy \(f(\kappa) = \kappa^2\)
associated with the bending of a curve, the Euler elastica,\(^1\) the Sadowski functional
\(f(\kappa) = (\kappa^2 + \tau^2)^2/\kappa^2\) related with bending of elastic developable surfaces \(^2\)
etc.

The second term in the model \(^1\) is \(H_b = \frac{1}{2} \int\!\!\!\!\!\int ds\, h(r)\) with \(r = |X(s) - X(s')|\),
represents the energy of the curve in the presence of long range forces, electrostatic interaction \(h(r) = 1/r\) as well as the screened Coulomb interaction (Yukawa) \(h(r) = e^{-\alpha r}/r\) and the Lennard-Jones potential \(h(r) = (\frac{\tau}{\kappa})^2 - (\frac{\tau}{\kappa})^6\) are some relevant possibilities to explore. The invariance under euclidean motions as well as reparametrizations of the model can be exploited to obtain several relevant
properties.

In order to obtain the shape equations we have to consider an infinitesimal deformation
of the energy \(\delta H\), as a result of deformations in the embedding functions, \(X \rightarrow X + \delta X\). For instance, given that the arclength is defined by
\(ds^2 = dX \cdot dX\), its deformation can be written as \(\delta ds = d(\delta X) \cdot t\). Thus, and following reference \(^3\),
we can see that an arbitrary infinitesimal deformation can be written as
\[
\delta H_a = \int ds \, \delta E_i \cdot n_i + \int ds \, Q,
\] (3)

where \(E_i\) (\(i = 1,2\)), are the Euler-Lagrange operators and \(Q\) the corresponding
Noether charge.

Let us now consider, the deformation of the second term in equation \(^1\). Since it
is symmetric under the change of \(s\) and \(s'\), we have that \(\delta H_b = \int\!\!\!\!\!\int ds\,\delta h(r) + \int ds\,\delta X \cdot \nabla \int ds\, h(r)\), where \(\nabla\) denotes the 3D gradient operator. After integration by parts and introducing the potential \(\Phi := \int ds\, h(r)\) we can write
\[
\delta H_b = \int d(\delta X \cdot t \, \Phi) - \int ds \, \kappa \delta X \cdot n_1 \Phi - \int ds \, \delta X \cdot t \, \Phi + \int ds \, \delta X \cdot \nabla \Phi.
\] (4)

Notice that the last term of equation \(^4\) can be decomposed in its normal and
tangential projections as \(\delta X \cdot \nabla \Phi = \delta X \cdot t \, \Phi + (\delta X \cdot n_i) n_i \cdot \nabla \Phi\). The first term
cancels the corresponding tangential projection in equation \(^4\). As expected by
invariance of \(H\) under reparametrizations, tangential deformations do not play a
role in the shape equations, these can be obtained from \(\delta H = \delta H_a + \delta H_b = 0\):
\[
\begin{align*}
E_1 &:= E_1 - \kappa \Phi + n_1 \cdot \nabla \Phi = 0, \\
E_2 &:= E_2 + n_2 \cdot \nabla \Phi = 0.
\end{align*}
\] (5)

The normal projections of the external force \(\nabla \Phi\), have contribution to these
equations as it should be. It is interesting to note that the potential \(\Phi\), contributes
directly to the shape equation along the principal normal direction \( n_1 \). This might be interpreted as a minimal coupling between the curve and the external field.

### 2.2. Translational invariance

By adding equations (3) and (4), the total deformation of the energy \( H \) can be cast as

\[
\delta H = \int ds \, E_i \delta X \cdot n_i + \int ds \, \dot{Q}
\]

where now, \( Q = Q + \delta X \cdot t \Phi \), is the corresponding Noether charge. According to equation (6), under a translation \( \delta X = a \), the energy is deformed as

\[
\delta H = a \cdot \int ds \left( E_i n_i - \dot{G} \right)
\]

where \( G = F - t \Phi \), being the relation with the elastic forces through \( Q = -a \cdot F \). In equilibrium \( G \) is a conserved vector field along the curve. Expressing in the local basis, \( G = G_\parallel t + G_i n_i \), the balance equation \( E_i n_i = G \) yields

\[
\dot{G}_\parallel - \kappa G_1 = 0,
\quad \dot{G}_1 + \kappa G_\parallel - \tau G_2 = E_1,
\quad \dot{G}_2 + \tau G_1 = E_2,
\]

the first of them, or Bianchi identity, implies that the arclength of the curve is preserved under the total force \( G \). In terms of the elastic and external forces we have

\[
\dot{F}_\parallel - \kappa F_1 = t \cdot \nabla \Phi,
\quad \dot{F}_1 + \kappa F_\parallel - \tau F_2 = E_1 + n_1 \cdot \nabla \Phi,
\quad \dot{F}_2 + \tau F_1 = E_2 + n_2 \cdot \nabla \Phi.
\]

The first equation tells us that arclength is not preserved under the elastic force \( F \), but it is balanced with the tangential component of the external force \( \nabla \Phi \). In the same way, this force contributes to the equilibrium in the normal directions as we can see from the last two equations in (8).

### 2.3. Rotational invariance

The invariance under rotations can be explored in a similar way as in ref. \[13\]. Considering an infinitesimal rotation \( \delta X = \Omega \times X \) in equation (6), we obtain

\[
\delta H = \Omega \cdot \int ds \left( E_i X \times n_i - \dot{M} \right),
\]

here, the relation \( Q = -\Omega \cdot M \), was used. The equation \( E_i X \times n_i = \dot{M} \), follows from rotational invariance. \( M \) is conserved along curves in equilibrium and it is identified with the torque respect to the origin. The long range potential contributes to the orbital torque as

\[
M = X \times (F - t \Phi) + T,
\]
where the local torque $T$ satisfies that $\dot{T} = G \times t$. Moreover, $\dot{T} = F \times t$ since the contribution of the external force is in the direction of the tangent vector $t$. Therefore, decomposing $T$ in the local basis, it satisfies

$$\dot{T} - \kappa T_1 = 0,$$

$$\dot{T}_1 - \tau T_2 + \kappa T_\parallel = F_2,$$

$$\dot{T}_2 + \tau T_1 = -F_1,$$

the external long range potential $\Phi$, does not contribute.

### 2.4. Elastic curves

Let us focus on the case where the elastic density energy is only function of the curvature $f = f(\kappa)$. The two corresponding Euler-Lagrange operators and the Noether charge are given by

$$E_1 = \ddot{f}_\kappa + (\kappa^2 - \tau^2)f_\kappa - \kappa(f + \Phi) + n_1 \cdot \nabla \Phi,$$

$$E_2 = 2\tau f_\kappa + \dot{f}_\kappa + n_2 \cdot \nabla \Phi,$$

$$Q = (f + \Phi)\Psi_\parallel + f_\kappa \Psi_1 - \dot{f}_\kappa \Psi_1 - 2\tau f_\kappa \Psi_2,$$

where $f_\kappa = \partial f / \partial \kappa$ and the projections of the deformation, $\delta X = \Psi_\parallel t + \Psi_i n_i$, were used. The conserved force $G$, is given by

$$G = (f_\kappa \kappa - f - \Phi)t + \dot{f}_\kappa n_1 + \tau f_\kappa n_2,$$

thus, the external potential $\Phi$ contributes to the tension in the curve. There is no contribution of the external force to the local torque $T$ and then $T = -f_\kappa n_2$.

Furthermore, we can write a first integral of the problem, in terms of the conserved quantities $G$ and $J = G \cdot T$:

$$G^2 = \dot{f}_\kappa^2 + (f_\kappa \kappa - f - \Phi)^2 + \frac{J^2}{f_\kappa^2}.$$  

The presence of the long range force, into the effective potential $V(\kappa, \Phi) = (f_\kappa \kappa - f - \Phi)^2$, does not allow their interpretation as a central potential $V(\kappa)$ as in the elastic case.

### 2.5. The Euler elastica

In the case of plane curves with bending energy $f = \kappa^2/2$, the Euler-Lagrange equation reduces to:

$$\ddot{\kappa} + \kappa \left( \frac{\kappa^2}{2} - \Phi \right) = -n \cdot \nabla \Phi,$$

this was found in a dynamical context in ref. The corresponding first order integral equation can be written as

$$\dot{\kappa}^2 + \left( \frac{\kappa^2}{2} - \Phi \right)^2 = G^2.$$
and for the particular case of electrostatic interaction it is given by
\[
\kappa^2 + \left( \frac{\kappa^2}{2} - \int \frac{ds'}{|X(s)-X(s')|} \right)^2 = G^2,
\]
in terms of the embedding function \(X(s)\), it is an integro-differential equation.

3. Surfaces

3.1. Shape equation

The geometric description of the surface is constructed considering that it is embedded in \(\mathbb{R}^3\). We parameterized it by local coordinates \(\xi^a\), \(a = 1, 2\), through \(x = X(\xi^a)\). Now there are two tangent vectors \(e_a = \partial_a X\) to the surface, and the unit normal vector field \(n\) is defined by \(e_a \cdot n = 0\) and \(n \cdot n = 1\). The induced metric on the surface is defined by the symmetric tensor \(g_{ab} = e_a \cdot e_b\), whose inverse is denoted \(g^{ab}\). We denote \(\nabla_a\) the covariant derivative compatible with the induced metric.

The model of the surface that we consider is the sum of two terms, a natural generalization of equation (1) for curves
\[
\mathcal{H} = \int dA f(K) + \frac{1}{2} \int dA \int dA' h(r),
\]
where \(dA = \sqrt{g} \, d^2 \xi\) is the infinitesimal area element on the surface, \(g\) being the determinant of the induced metric.

The first term in equation (20), \(\mathcal{H}_1\), is the elastic energy of the surface; \(f(K)\) is a scalar function constructed with the geometry of the surface. \(K = g^{ab} K_{ab}\) the mean curvature in terms of the second fundamental form \(K_{ab} = -\nabla_a e_b \cdot n\). The second term in equation (20), \(\mathcal{H}_2\), involves the external potential \(h(r)\), similar to the case of curves, section 2. The functional \(\mathcal{H}\), is invariant under reparameterizations and euclidean motions of the surface.

Given the invariance under reparameterizations of the energy, only normal deformations \(\delta X \cdot n\) play a role in the Euler-Lagrange equations. Deforming the first term of the energy (20), we can obtain
\[
\delta \mathcal{H}_1 = \int dA \, E \, \delta X \cdot n + \int dA \nabla_a Q^a, \tag{21}
\]
where \(E\) is the Euler-Lagrange derivative of the elastic model and \(Q^a\) the corresponding Noether charges. The remaining deformation of the energy \(\mathcal{H}\) in eq.(20) involves the external potential. We obtain
\[
\delta \mathcal{H}_2 = \int dA \left( K \Phi + n \cdot \nabla \Phi \right) \delta X \cdot n + \int dA \nabla_a \left( g^{ab} \delta X \cdot e_b \Phi \right), \tag{22}
\]
where we have defined \(\Phi = \int dA' h(r)\). The shape equation can be found by setting \(\delta \mathcal{H}_1 + \delta \mathcal{H}_2 = 0\). From equations (21) and (22) we have that the Euler-Lagrange derivative is
\[
\mathcal{E} = E + K \Phi + n \cdot \nabla \Phi, \tag{23}
\]
in equilibrium, the elastic forces are balanced with the external long range force, \( E = -K \Phi - n \cdot \nabla \Phi \). In the case of minimal surfaces where \( K = 0 \), there is no contribution of the second term to the shape equation.

### 3.2. Translational and rotational invariance

We can write the total deformation of the energy in the form

\[
\delta \mathcal{H}' = \int dA \left( E + K \Phi + n \cdot \nabla \Phi \right) \delta \mathbf{X} \cdot \mathbf{n} + \int dA \nabla_a \left( Q^a + g^{ab} \delta \mathbf{X} \cdot \mathbf{e}_b \Phi \right).
\]

By doing an infinitesimal translation \( \delta \mathbf{X} = \mathbf{a} \) we obtain that

\[
\delta \mathcal{H}' = \mathbf{a} \cdot \int dA \left[ (E + K \Phi + n \cdot \nabla \Phi) \mathbf{n} - \nabla_a (f^a - e^a \Phi) \right],
\]

(25)

where the relation between the Noether charge and the elastic forces, \( Q^a = -\mathbf{a} \cdot f^a \), was used. Thus, on a surface in equilibrium, \( h^a = f^a - e^a \Phi \) is a conserved vector field, as a consequence of invariance under translations. Using the decomposition in the local basis, \( f^a = f^{ab} \mathbf{e}_b + f^a \mathbf{n} \), we can write

\[
\nabla_a f^{ab} + K_b f^a = \nabla^b \Phi,
\]

\[
\nabla_a f^a - K_{ab} f^{ab} = E + n \cdot \nabla \Phi,
\]

(26)

comparing with eq.(24) we can see that they match for the case of curves.

Let us see the consequences of the invariance of the energy under rotations. Let \( \delta \mathbf{X} = \Omega \times \mathbf{X} \) be an infinitesimal rotation in the deformation of the energy equation (24). We have

\[
\delta \mathcal{H}' = \Omega \cdot \int dA \left[ \mathcal{E} \mathbf{X} \times \mathbf{n} - \nabla_a M^a \right],
\]

(27)

the invariance under rotations of the energy implies that \( \mathcal{E} \mathbf{X} \times \mathbf{n} = \nabla_a M^a \). Thus, in equilibrium \( M^a \) is a conserved vector field on the surface and it is identified with the torque. We can decompose this vector in an orbital torque plus a local one, in the form \( M^a = \mathbf{X} \times (f^a - e^a \Phi) + s^a \) where the local torque satisfies \( \nabla_a s^a = f^a \times e_a \).

### 3.3. Lipid membranes

In this case the energy is quadratic in the mean curvature \( f(K) = \frac{K^2}{2} \). The Euler-Lagrange derivative and the conserved Noether charge of the elastic energy are modified as follows

\[
\mathcal{E} = -\nabla^2 K + K \left( 2K_G - \frac{1}{2} K^2 \right) + K \Phi + n \cdot \nabla \Phi
\]

\[
h^a = K \left( 2K_{ab} - K g^{ab} \right) \mathbf{e}_b - 2\nabla^a K \mathbf{n} + \Phi \mathbf{e}^a
\]

(28)

where \( 2K_G = \mathcal{R} \), is the curvature scalar of the surface.

It is possible to impose the relations (2) directly in the energy (1) by considering them as constraints of the variational problem, and then fixing them through the introduction of indeterminate Lagrange multipliers. In this way, the curvature $\kappa$ and $X$ can be varied independently. In order to confine the curve into a surface $x = Y(\xi)$ we have to add in equation (1) this constraint in the form

$$H_c = H + \int ds \lambda \cdot [X(s) - Y(\xi)].$$ (29)

As in ref. 3 we take the variation with respect to $X$ obtaining the following relation

$$\delta X H_c = \int ds (\dot{\mathbf{F}} + \lambda + \nabla \Phi) \cdot \delta X,$$ (30)

where $\nabla \Phi$ denotes the euclidean gradient, and the multiplier $F$ is given by

$$F = (-\lambda T - \kappa f_\tau) t + \left(\frac{\tau}{\kappa} f_\tau + \dot{f}_\kappa\right) n_1 + \left(\tau f_\kappa - \kappa f_\tau - \frac{d}{ds} \left(\frac{\dot{f}_\tau}{\kappa}\right)\right) n_2,$$ (31)

$\lambda T$ being the Lagrange multiplier that enforces the curve to be parameterized by its arclength. From equation (30), we see that along a curve in equilibrium, $\dot{\mathbf{F}} + \nabla \Phi = -\lambda$. We can also see that $\lambda$ is orthogonal to the surface and therefore we can write, $\lambda = -\lambda n$. As a consequence the tangential projection vanishes identically, i.e. $(\dot{\mathbf{F}} + \nabla \Phi) \cdot e_a = 0$, in particular, projection along the tangent $t$ determines the multiplier $\lambda T$ in equation (31) to be:

$$\lambda T = f + \Phi - 2\kappa f_\kappa - \tau f_\tau + \sigma,$$ (32)

where $\sigma$ is a constant that tell us that the length of the curve is fixed. Thus, the external potential $\Phi$ contributes to the effective tension. Projections of $\dot{\mathbf{F}} + \nabla \Phi$, along the normal vectors $n_i$ give the Euler-Lagrange equations. We have

$$\left(\dot{\mathbf{F}} + \nabla \Phi\right) \cdot n_1 = \ddot{f}_\kappa + (\kappa^2 - \tau^2) f_\kappa + 2\tau \frac{d}{ds} \left(\frac{\dot{f}_\tau}{\kappa}\right) + \dot{\tau} \left(\frac{\dot{f}_\tau}{\kappa}\right) + 2\kappa \tau f_\tau$$

$$- \kappa (f + \Phi + \sigma) + n_1 \cdot \nabla \Phi,$$ (33)

$$\left(\dot{\mathbf{F}} + \nabla \Phi\right) \cdot n_2 = 2\frac{d}{ds} (\tau f_\kappa) - \dot{\tau} f_\kappa - d^2 \frac{d}{ds^2} \left(\frac{\dot{f}_\tau}{\kappa}\right) - \frac{d}{ds} (\kappa f_\tau)$$

$$+ \frac{\tau^2 \dot{f}_\kappa}{\kappa} + n_2 \cdot \nabla \Phi,$$ (34)

these equations reproduce the $E_i$ derivatives of equations (5) in the case of curves without restriction. Then we can write $(\dot{\mathbf{F}} + \nabla \Phi) = E_n n_i$. If we identify the normal to the surface to be the principal normal of the curve, $n = n_1$, then we can write

$$\dot{\mathbf{F}} + \nabla \Phi = E_n n + E_1 l,$$ (35)
where the unit vector field \( l = n_2 \) is defined by the conditions \( l \cdot t = 0 \) and \( l \cdot n = 0 \).

From equation (33), the Euler Lagrange equation, \( E_l = 0 \) and the normal force, \( -\lambda \), can be recognized as

\[
E_l = (\dot{F} + \nabla \Phi) \cdot l, \quad \quad \quad \quad (36)
\]

\[
-\lambda = (\dot{F} + \nabla \Phi) \cdot n. \quad \quad \quad \quad (37)
\]

That is, the long range force contributes to the normal geometric forces adding the terms \( \kappa \Phi - n \cdot \nabla \Phi \).

With this formulation one can also consider the long range interaction between points on the surface with points on the curve, this can be achieved by adding the term

\[
\iint ds \, dA' \, h(R), \quad (38)
\]

where now \( R = |X(s) - Y(\xi^a(s'))| \). This term considers the contributions to the interaction of the surface on the curve, for every point on the curve. Notice that the back-reaction that the curve has on the surface is not considered. The force satisfies

\[
\dot{F} + \nabla \left( \Phi + \tilde{\Phi} \right) = -\Lambda(s), \quad (39)
\]

where \( \tilde{\Phi} = \int dA' \, h(R) \) is the contribution of the potential on the surface. Bearing in mind that now we can vary with respect to the coordinates on the surface \( \xi^a \), that leads us to

\[
\delta \xi \mathcal{H} = -\int ds \, \Lambda(s) \cdot e_a \delta \xi^a - \int ds \, \nabla \tilde{\Phi} \cdot e_a \delta \xi^a, \quad (40)
\]

such that \((\Lambda(s) + \nabla \tilde{\Phi}) \cdot e_a = 0\). This means that the only force that appears on the surface is \( \nabla \Phi \).

From the variation of \( \mathcal{H} \) with respect to the remaining variables one can obtain the Lagrange multipliers:

\[
\Lambda_\kappa = \mathcal{H}_\kappa, \quad \quad \Lambda_\tau = \mathcal{H}_\tau. \quad \quad \quad \quad (41)
\]

Using the properties of stationarity and Frenet-Serret equations, the following relation for the components of the multiplier \( F \), which corresponds to the force, can be derived:

\[
F = (-\lambda - \kappa \mathcal{H}_\kappa) \, t + \left( \frac{\mathcal{H}_\tau}{\kappa} + \mathcal{H}_\kappa \right) \, n_1 + \left( \tau \mathcal{H}_\kappa - \kappa \mathcal{H}_\tau - \frac{d}{ds} \left( \kappa^{-1} \mathcal{H}_\tau \right) \right) \, n_2, \quad (42)
\]

where only the multiplier \( \lambda \) is missing. To find \( \lambda \) one can take the derivative of \( F \), and compare its components with those obtained in \( (39) \), which are clearly different.
form zero. With this, one can find the following relations:

\[
\begin{align*}
\dot{F} \cdot t &= - (\kappa H - \kappa H) \kappa - \lambda = -\Phi, \\
\dot{F} \cdot n_1 &= \ddot{H} - (\kappa^2 + \tau^2) H + 2\tau \left( \kappa^{-1} H \right) + \tau \left( \kappa H \right) + \kappa \tau H - \kappa \lambda \\
&= - \left( \nabla \Phi + \Lambda + \nabla \Phi \right) \cdot n_1, \\
\dot{F} \cdot n_2 &= (\tau H - \kappa H) \kappa - (\kappa H) \kappa - \tau \left( \kappa^{-1} H \right) + \tau \left( \frac{\tau}{\kappa} \dot{H} + \dot{H} \right) \\
&= - \left( \nabla \Phi + \Lambda + \nabla \Phi \right) \cdot n_2.
\end{align*}
\] (43)

From (43) one can integrate to obtain \( \lambda \), noticing that this expression is a total derivative on \( s \):

\[
\lambda = H - 2\kappa H - \tau H + \Phi + \sigma,
\] (46)

where \( \sigma \) again fixes the length of the curve. With (46), the multiplier \( F \) is determined as:

\[
F = (\kappa H + \tau H - \dot{H} - \Phi - \sigma) t + \left( \frac{\tau}{\kappa} H + \dot{H} \right) n_1 \\
+ \left( \tau H - \kappa H - \left( \kappa^{-1} H \right) \right) n_2.
\] (47)

Amusingly, both the force that keeps the curve on the surface and the interaction with the membrane, do not appear in this expression; only the potential due to the interaction of the curve itself appears in the tangential component.

5. Concluding Remarks

The interest on studying charged polymers and surfaces comes because in nature, there are some systems, such as the polyelectrolytes, that can be modeled as charged chains which require a description involving long-range electrostatic forces.

In this work we have studied curves and surfaces under the influence of long-range and elastic forces, both from the classical differential geometry perspective. Besides the usual elastic terms, we introduce the long-range interactions as an integral of the corresponding potential, which takes into account the interaction between all points of the curve. The corresponding shape equations were obtained form a variational principle, considering small deformations.

In the case of curves the resulting Euler-Lagrange equations can be written as the sum of the usual elastic term and normal projections of the external force. Amusingly, in one normal directions there are a contribution of the bare potential, which further is coupled with the curvature. We can interpret this term in (5) as a minimal coupling. Given the translational and rotational invariance of the model, we were able to write the equations for the force and torque, noticing that the long-range potential \( \Phi \) contributes directly to the orbital torque, unlike the force and the local torque, where it only contributes through its gradient. In the case where the elastic force is quadratic on the curvature, the well known Euler elastica
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model, the corresponding equation for $\kappa$ becomes an integro-differential equation that will be solved numerically elsewhere. For the charged surfaces we perform the same analysis straightforwardly, and found a similar minimal coupling, now with the mean curvature, and the projection of the long-range force in the normal direction; both in the Euler-Lagrange derivative.

We also consider the case of a curve constrained to lie on a surface, this analysis was done by introducing Lagrange multipliers for the constraints of the system, recovering the previous results, and obtaining additional information for the normal force on the curve. Moreover, with this formalism, we could write the corresponding interaction in the case where both the surface and the curve were charged.

To integrate all the resulting equations, numerical methods must be used, although it was not the main task of this work, we are in the process of implementing this in a subsequent study.

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