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Bounds on the acyclic disconnection of a digraph

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Abstract
The *acyclic disconnection* $\overrightarrow{\omega}(D)$ of a digraph D is the maximum possible number of (weakly) connected components of a digraph obtained from D by deleting an acyclic set of arcs. In this paper, we provide new lower and upper bounds in terms of properties such as the degree, the directed girth, and the existence of certain subdigraphs and bounds for bipartite digraphs, p-cycles, and some circulant digraphs. Finally, as a consequence of our bounds, we prove the Conjecture of Caccetta and Häggkvist for a particular class of digraphs.

Keywords Acyclic · Acyclic disconnection · Girth · Bipartite digraphs · p-cycles

1 Introduction

In 1999, Neumann-Lara [1] defined the acyclic disconnection of a digraph as a measure of the complexity of the cyclic structure. The *acyclic disconnection* $\overrightarrow{\omega}(D)$ of a digraph D is the maximum possible number of (weakly) connected components of a digraph obtained from D by deleting an acyclic set of arcs. Equivalently, the acyclic disconnection can be defined in terms of vertex colorings, cycle transversals, or certain subdigraphs [1, 2], in particular, as the maximum number of colors in a vertex coloring of D not producing *proper directed cycles* that is a cycle where every pair of adjacent vertices have different colors.

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In [3], it was proved that the problem of determining $\overrightarrow{\omega}(D)$ of an arbitrary digraph D is NP-complete. The acyclic disconnection of a digraph has been studied in different classes of tournaments [1, 2, 4–7], and it has been related to other invariants such as the maximum order of an acyclic subset of vertices, $\overrightarrow{\beta}(D)$, or the number of vertices of the digraph D [1], the dichromatic number (introduced by Neumann-Lara in 1982) [1, 7], the Feedback Arc Set [3], and the girth [8].

Let $i_1, i_2, \ldots, i_d \in \mathbb{Z}_n \setminus \{0\}$. A circulant digraph $\overrightarrow{C}_n(i_1, i_2, \ldots, i_d)$ has vertex set the elements of $\mathbb{Z}_n$, and (a, b) is an arc if and only if $b = a + i_j$ for some $i_j \in \{i_1, i_2, \ldots, i_d\}$, where the sum is taken in $\mathbb{Z}_n$. We use the book [9] for terminology and definitions not given here.

Lower bounds on the acyclic disconnection in terms of $\overrightarrow{\beta}(D)$ were established in [8]. In particular, the following theorem was stated.

Theorem 1 [8] *Every digraph D with girth $g \geq 4$ that contains a subdigraph isomorphic to an acyclic tournament of order k has $\overrightarrow{\omega}(D) \geq k + g - 3$.*

In this paper, we give new bounds on the acyclic disconnection of digraphs. We present lower bounds in terms of the existence of certain subdigraphs and lower bounds for the p-cycles and certain kinds of circulant digraphs. We present upper bounds in terms of the order, the degree, and the directed girth and upper bounds for r-regular bipartite digraphs, p-cycles. Finally, as a consequence of our bounds, we prove the Conjecture of Caccetta and Häggkvist for a particular class of digraphs.

2 Bounds on acyclic disconnection

Let Γ_s denote the set of colors $\{c_1, c_2, \ldots, c_s\}$. Let D be a digraph and $\varphi : V(D) \to \Gamma_s$ a vertex coloring of D. For every $c \in \Gamma_s$, the *chromatic class* corresponding to color c is the set of vertices $V_c \subseteq V(D)$, such that $\varphi(v) = c$ for all $v \in V_c$. Throughout this work, we also use the term chromatic class to refer to the subdigraph induced by the set of vertices in the chromatic class. The color c_α is a *s*ingular class of φ if there is $u \in V(D)$, such that $\varphi(u) = c_\alpha$ and $\varphi(v) \neq c_\alpha$ for every $v \in V(D) \setminus \{u\}$. We say that a subdigraph H of D is *proper colored* if $\varphi(u) \neq \varphi(v)$ for any two vertices $u, v \in V(H)$, such that $uv \in A(D)$. Therefore, a proper (colored) cycle is a cycle, such that any two adjacent vertices u, v on the cycle have different color. The set of external arcs of a coloring $\varphi : V(D) \to \Gamma_s$ is the arc set $\{uv \in A(D) : \varphi(u) \neq \varphi(v)\}$. The *heterochromatic digraph* $H_\varphi(D)$ is the spanning subdigraph of D with arc set $\{uv \in A(D) : \varphi(u) \neq \varphi(v)\}$ [2]. Observe that a vertex coloring φ is *externally acyclic* if $H_\varphi(D)$ is an acyclic digraph.

As we mention in the Introduction, Neumann-Lara [1] defined the *acyclic disconnection* $\overrightarrow{\omega}(D)$ of a digraph D, as the maximum possible number of connected components of adigraph obtained from D by deleting an acyclic set of arcs. Equivalently, the *acyclic disconnection* $\overrightarrow{\omega}(D)$ can be defined as the maximum number of colors in a vertex coloring of D not producing proper (directed) cycles. Our objective in this section is to establish upper bounds on this parameter.

Let D be a digraph and F be a subdigraph of D. A vertex $v \in V(F)$ is *interior* in F if $N^+(v) \subseteq V(F)$ or $N^-(v) \subseteq V(F)$. The set of interior vertices $I(F)$ of F is the union of $I^+(F)$ or $I^-(F)$, where $I^\epsilon(F) = \{v \in V(F) : N^\epsilon(v) \subseteq V(F)\}, \epsilon \in \{-, +\}$.

Lemma 1 *Let D be a digraph and R a subset of vertices, such that $D[R]$ is an acyclic subdigraph. If every vertex $b \in V(D) \setminus R$ is an interior vertex of $D - R$, then $\overrightarrow{\omega}(D) \geq |R| + 1$.*

Proof Let $R = \{x_1, x_2, \ldots, x_{|R|}\}$. Let φ be a coloring, such that $\varphi(v) = i$ if $v = x_i$, $i = 1, 2, \ldots, |R|$, and $\varphi(v) = |R| + 1$ if $v \notin R$. Let γ be a cycle of D. Clearly, there exists $v \in V(\gamma) \setminus R$ and by the hypothesis $v \in I^-(D-R) \cup I^+(D-R)$. Suppose that $v \in I^-(D-R)$, then γ has the arc $v'v$ with $v' \in V(D-R)$; thus, γ has two adjacent vertices of the same color. We reason analogously if $v \in I^+(D-R)$. Therefore, φ is an external acyclic coloring and $\overrightarrow{\omega}(D) \geq |R| + 1$. □

Theorem 2 (i) *Let $D \cong \vec{C}_n(1, 2, \ldots, k)$ be a circulant digraph. Then, $\overrightarrow{\omega}(D) = 1$ if $n \leq 2k$; and $\overrightarrow{\omega}(D) \geq n - 2k + 1$ if $n \geq 2k + 1$.*

(ii) *Let $D \cong \vec{C}_{2n}(1, 3, \ldots, 2k-1)$ be a circulant digraph. Then, $\overrightarrow{\omega}(D) \geq 2n - 2k - 1$ if $2n \geq 4k$.*

Proof Let φ be an external acyclic coloring.

(i) When $n \leq 2k$ and n odd, this circulant digraph contains a symmetric Hamiltonian cycle; thus, every vertex must have the same color. And if n is even, then all the diagonals $\{i, i+k\}$ of this circulant digraph are symmetric. Since φ is an external acyclic coloring, both vertices of $\{0, k\}$ must have the same color, say r_1, and both vertices of $\{i, i+k\}$, $i < k$, have color r_2. If $r_1 \neq r_2$, then the cycle $(0, i, k, i+k, 0)$ has not two adjacent vertices with the same color which is a contradiction. Therefore, $\overrightarrow{\omega}(D) = 1$ if $n \leq 2k$. Next, suppose that $n \geq 2k + 1$. Let $R = \{0, 1, \ldots, n-2k-1\}$. Clearly, $D[R]$ is acyclic. Let i with $n - 2k \leq i \leq n - k - 1$. Then, every $j \in N^+(i)$ satisfies that $j \leq n - 1$, and hence, $N^+(i) \subset D - R$ or equivalently $i \in I^+(D-R)$. Analogously, if $n - k \leq i \leq n - 1$, then every $j \in N^-(i)$ satisfies that $j \geq n - 2k$, and therefore, $i \in I^-(D-R)$. By Lemma 1, it follows that $\overrightarrow{\omega}(D) \geq |R| = n - 2k + 1$, and item (i) is proved.

(ii) Let $R = \{0, 1, \ldots, 2n-2k-1\}$. Clearly, $D[R]$ is acyclic and reasoning as in item (i) we obtain the desired result.

□

Lemma 2 *Let D be a digraph of order n with $\overrightarrow{\omega}(D) \geq 2$. Every external acyclic coloring of $V(D)$ has at least two chromatic classes C and C', such that $|I^\epsilon(C)| \geq 1$ and $|I^\epsilon(C')| \geq \delta^\epsilon(D) + 1 - |I^\epsilon(C)|$ for $\epsilon \in \{+, -\}$.*

Proof Let φ be an external acyclic coloring of D which has at least two colors, because $\overrightarrow{\omega}(D) \geq 2$. Let H_φ be the corresponding acyclic subdigraph. Then, there exists a vertex v in H_φ, such that $d^+_{H_\varphi}(v) = 0$. Let C be the chromatic class of φ, such that $v \in V(C)$, then $|I^+(C)| \geq 1$. If there is a chromatic class C' different from C

such that $I^+(C') \neq \emptyset$, then the lemma holds. Therefore, we suppose $I^+(C') = \emptyset$ for all C' different from C. Let $D' = H_\varphi - I^+(C)$ and consider $u \in V(D')$ with $d^+_{D'}(u) = 0$. Then, $u \in V(C')$, for some $C' \neq C$ and $N^+(u) \subseteq I^+(C) \cup V(C')$. Then, $|V(C')| \geq \delta^+(D) + 1 - |I^+(C)|$. Analogously, for $|V(C')| \geq \delta^-(D) + 1 - |I^-(C)|$, we use that there exists a vertex v in H_φ, such that $d^-_{H_\varphi}(v) = 0$. □

Shen proved the following result.

Theorem 3 [10] *Every digraph D of order n and $\delta^+(D) \geq (3-\sqrt{7})n$ (or $\delta^-(D) \geq (3-\sqrt{7})n$) contains a directed triangle.*

Lemma 3 *Let D be a digraph with $d = \max\{\delta^+, \delta^-\}$ and let F, F' be two subdigraphs, such that $F \subset F'$ and for every $u \in V(F)$, it follows that $u \in I^+(F')$ if $d = \delta^+$ or $u \in I^-(F')$ if $d = \delta^-$. Then*

(i) $|V(F')| \geq |V(F)| + d - |A(F)|/|V(F)|$ *where $A(F)$ is the set of arcs of F.*
(ii) $|V(F')| \geq d + (\sqrt{7}-2)|V(F)|$*, if $g \geq 4$, (where g is the girth of D.)*
(iii) $|V(F')| \geq \min\{2d, |V(F)| + d\}$*, if D is bipartite.*

Proof Assume that $d = \delta^+$.

(i) Notice that there exists $v_0 \in V(F)$, such that $d^+_F(v_0) \leq |A(F)|/|V(F)|$, yielding that $|N^+(v_0) \cap V(F'-F)| \geq \delta^+ - |A(F)|/|V(F)|$ because $u \in I^+(F')$ for all $u \in V(F)$. Thus, $|V(F')| \geq |V(F)| + \delta^+ - |A(F)|/|V(F)|$ and item (i) holds.
(ii) As a consequence of Theorem 3, it follows that if $g \geq 4$, there exists $v_0 \in V(F)$, such that $d^+_F(v_0) < (3-\sqrt{7})|V(F)|$. Thus, $|N^+(v_0) \cap V(F'-F)| \geq \delta^+ - (3-\sqrt{7})|V(F)|$, yielding that $|V(F')| \geq |V(F)| + \delta^+ - (3-\sqrt{7})|V(F)| = \delta^+ + (\sqrt{7}-2)|V(F)|$ and item (ii) holds.
(iii) Let U, W be a bipartition of the vertices of D. If $V(F) \subseteq U$ (or $V(F) \subseteq W$), then $|V(F')| \geq |V(F)| + \delta^+$, and the result clearly holds. Otherwise, there are $u \in V(F) \cap U$ and $w \in V(F) \cap W$, $|V(F')| \geq 2\delta^+$. Thus, item (iii) also holds.

□

Theorem 4 *Let D be a digraph of order n, girth g and with $d = \max\{\delta^+, \delta^-\}$. Then, the acyclic disconnection*

(i) $\overrightarrow{\omega}(D) \leq n - d$.
(ii) $\overrightarrow{\omega}(D) \leq n - (3d-1)/2$ *if* $g \geq 3$.
(iii) $\overrightarrow{\omega}(D) \leq n + 1 - (\sqrt{7}-1)d$ *if* $g \geq 4$.
(iv) $\overrightarrow{\omega}(D) \leq n - 2d + 1$ *if D is bipartite.*

Proof Assume that $d = \delta^+$.

(i) Let φ be an external acyclic coloring of D using $\overrightarrow{\omega}(D)$ colors. If $\overrightarrow{\omega}(D) = 1$, then, since $d = \max\{\delta^+, \delta^-\} \leq n-1$, it follows that $\overrightarrow{\omega}(D) = 1 \leq n-d$. If $\overrightarrow{\omega}(D) \geq 2$, then by Lemma 2, there exists a chromatic class C, such that $|I^+(C)| \geq 1$, or equivalently $|V(C)| \geq \delta^+ + 1$. Since there are at most $n - |V(C)| + 1$ chromatic classes, we obtain
$$\overrightarrow{\omega}(D) \leq 1 + n - |V(C)| \leq n - \delta^+.$$

(ii) Since the girth $g \geq 3$, it follows that $\overrightarrow{\omega}(D) \geq 2$. Therefore, we can apply Lemma 2 yielding that there exists a chromatic class C, such that $|I^+(C)| \geq 1$. By Lemma 3, with $F' = C$ and F the induced subdigraph by $I^+(C)$, and taking into account that $|A(F)| \leq |V(F)|(|V(F)| - 1)/2$, because the girth $g \geq 3$, it follows that:

$$|V(C)| \geq |I^+(C)| + \delta^+ - \frac{|I^+(C)| - 1}{2} = \delta^+ + \frac{|I^+(C)| + 1}{2}. \tag{1}$$

If $|I^+(C)| \geq \delta^+$, then

$$\overrightarrow{\omega}(D) \leq 1 + n - |V(C)| \leq 1 + n - \frac{3\delta^+ + 1}{2} = n - \frac{3\delta^+ - 1}{2},$$

and the result holds. Then, we assume $|I^+(C)| \leq \delta^+ - 1$. From Lemma 2 and from (1), it follows that there exists $C' \neq C$, such that $|V(C')| \geq \delta^+ - |I^+(C)| + 1$. Then, we have

$$\begin{aligned} |V(C)| + |V(C')| &\geq \delta^+ + \frac{|I^+(C)| + 1}{2} + \delta^+ - |I^+(C)| + 1 \\ &= 2\delta^+ - \frac{|I^+(C)| - 1}{2} + 1 \\ &\geq \frac{3\delta^+}{2} + 2. \end{aligned}$$

Since there are at most $n - (|V(C)| + |V(C')|) + 2$ chromatic classes, therefore

$$\overrightarrow{\omega}(D) \leq 2 + n - (|V(C)| + |V(C')|) \leq n - \frac{3\delta^+}{2} < n - \frac{3\delta^+ - 1}{2}.$$

Hence, item (ii) holds.

(iii) Suppose that the girth $g \geq 4$. Therefore, $\overrightarrow{\omega}(D) \geq 2$. By Lemma 3 (ii), with $F' = C$ and F the induced subdigraph by $I^+(C)$, it follows that:

$$|V(C)| \geq \delta^+ + (\sqrt{7} - 2)|I^+(C)|. \tag{2}$$

If $|I^+(C)| \geq \delta^+$, we have $|V(C)| \geq (\sqrt{7} - 1)\delta^+$. Then, $\overrightarrow{\omega}(D) \leq 1 + n - |V(C)| \leq 1 + n - (\sqrt{7} - 1)\delta^+$ and the result holds. Hence, we continue the proof assuming that $|I^+(C)| < \delta^+$.

By Lemma 2, and by (2), we have

$$\begin{aligned} |V(C)| + |V(C')| &\geq (\delta^+ + (\sqrt{7} - 2)|I^+(C)|) + (\delta^+ - |I^+(C)| + 1) \\ &= 2\delta^+ - (3 - \sqrt{7})|I^+(C)| + 1 \\ &\geq (\sqrt{7} - 1)\delta^+ + 1. \end{aligned}$$

Therefore

$$\overrightarrow{\omega}(D) \leq 2 + n - (|V(C)| + |V(C')|) \leq n - (\sqrt{7} - 1)\delta^+ + 1.$$

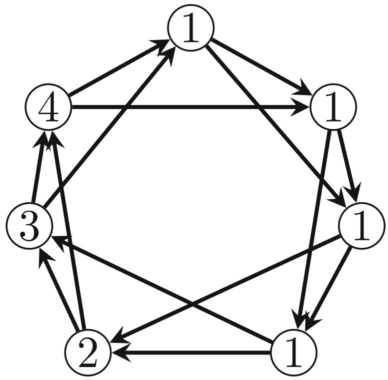

Fig. 1 $C_7(1,2)$

Hence, item (iii) holds.

(iv) Suppose that D is bipartite. By Lemma 3 (iii), it follows that:

$$|V(C)| \geq \min\{2\delta^+, \delta^+ + |I^+(C)|\}. \tag{3}$$

Hence, if $|I^+(C)| \geq \delta^+$, then $|V(C)| \geq 2\delta^+$ yielding that $\overrightarrow{\omega}(D) \leq 1 + n - |V(C)| \leq 1 + n - 2\delta^+$ and the result holds. Hence, by (3), we continue the proof assuming that

$$|I^+(C)| \leq \delta^+ - 1 \text{ and } |V(C)| \geq \delta^+ + |I^+(C)|.$$

By Lemma 2, we have

$$|V(C)| + |V(C')| \geq (\delta^+ + |I^+(C)|) + (\delta^+ - |I^+(C)| + 1) = 2\delta^+ + 1.$$

It follows that:

$$\overrightarrow{\omega}(D) \leq n + 2 - (|V(C)| + |V(C')|) \leq n - 2\delta^+ + 1.$$

Hence, the theorem holds.

□

As an immediate consequence of Theorem 2 and Theorem 4, we can write the following corollary.

Corollary 1 *For all* $n \geq 5$, $\overrightarrow{\omega}(\overrightarrow{C}_n(1,2)) = n - 3$. *See Fig. 1 for* $n = 7$.

Remark 1 The upper bound on $\overrightarrow{\omega}(D)$ given in Theorem 4 item (ii) is tight at least for $\delta^+ = 1, 2$, because for a directed cycle $\overrightarrow{\omega}(\overrightarrow{C}_n) = n - 1$ and by corollary.

A bipartite tournament is an oriented complete bipartite graph. Theorem 4 allows us to establish the following result for bipartite tournaments.

Corollary 2 *If* T *is an* r*-regular bipartite tournament of order* $4r$, *then* $\overrightarrow{\omega}(D) \leq 2r+1$.

The above result is also obtained in [2]. Moreover, this upper bound was shown to be tight for $\overrightarrow{C}_4[\overline{K}_r]$ also known as a complete p-cycle for $p = 4$.

A generalized p-cycle is a digraph D, such that its set of vertices can be partitioned in p parts,

$$V(D) = \cup_{\alpha \in \mathbb{Z}_l} V_\alpha,$$

in such a way that the vertices in the partite set V_α, are only adjacent to vertices in $V_{\alpha+1}$, where the sum is in $\mathbb{Z}_p$. If D is strongly connected, $N^+(V_\alpha) = V_{\alpha+1}$. Observe that bipartite digraphs are generalized p-cycles with $p = 2$. Gómez, Padró, and Perennes showed in [11] that a digraph is a generalized p-cycle if and only if for any pair of vertices u, v, the lengths of all paths from u to v are congruent modulo p. Hence, the girth of a p-cycle is at least p. Clearly, when $p \geq 3$ the transitive tournament contained in a p-cycle is an arc. As a consequence of Theorem 1 and Theorem 4 we obtain the following result.

Corollary 3 *Let D be a p-generalized cycle with $p \geq 3$ of order n and $d = \max\{\delta^+, \delta^-\}$. Therefore*

$$p - 1 \leq \overrightarrow{\omega}(D) \leq \begin{cases} n - 2d + 1 & \text{if } p \text{ even} \\ n - (3d-1)/2 & \text{if } p \text{ odd.} \end{cases}$$

In the next result, we improve the lower bound of the above corollary.

The number of weak components of a digraph D (i.e., the number of connected components of its underlying graph) is denoted by $\omega(D)$.

Proposition 5 *Let D be a p-generalized cycle of order n, $p \geq 3$ and partite sets $V_1, V_2, \ldots, V_p$. Then,*

$$\overrightarrow{\omega}(D) \geq n - \min\{|V_i| + |V_{i+1}| - \omega(D[V_i \cup V_{i+1}])\}.$$

Moreover, if $D[V_i \cup V_{i+1}]$ is weakly connected, then $\overrightarrow{\omega}(D) \geq n - \min\{|V_i| + |V_{i+1}|\} + 1$, and the equality is obtained when the p-cycle is complete.

Proof Consider two consecutive partite sets V_i and V_{i+1}. Clearly, $D - (V_i \cup V_{i+1})$ is acyclic and every vertex $b \in V_i \cup V_{i+1}$ is an interior vertex of $D[V_i \cup V_{i+1}]$. If $\omega(D[V_i \cup V_{i+1}]) = k$, then we can color each vertex of $D - (V_i \cup V_{i+1})$ with a different color and the vertices each component of $D[V_i \cup V_{i+1}]$ with the same color. Thus, $\overrightarrow{\omega}(D) \geq n - (|V_i| + |V_{i+1}|) + k$. Hence, $\overrightarrow{\omega}(D) \geq n - \min\{|V_i| + |V_{i+1}| - \omega(D[V_i \cup V_{i+1}])\}$. If D is a complete p-cycle, then $D[V_i \cup V_{i+1}]$ is weakly connected and $\overrightarrow{\omega}(D) \geq n - \min\{|V_i| + |V_{i+1}|\} + 1$. Moreover, every external acyclic coloring must have two consecutive partite sets colored with the same color, because the p-cycle is complete. Hence, $\overrightarrow{\omega}(D) \leq n - \min\{|V_i| + |V_{i+1}|\} + 1$ and the result follows. □

As a consequence of Theorem 1, and using Theorem 4, the following corollary is direct.

Corollary 4 *Let D be a digraph on n vertices, girth $g \geq 4$, minimum out-degree $\delta^+ \geq 1$ that contains a subdigraph isomorphic to an acyclic tournament of order k. Then*

(i) $g \leq \overrightarrow{\omega}(D) - k + 3$.
(ii) $g \leq n - k + 4 - (\sqrt{7} - 1)\delta^+$.
(iii) $g \leq n - 2\delta^+ + 4 - k$ *if D is bipartite.*

A (d, g)-digraph is a d-regular digraph with girth g. Behzand, Chartrand, and Wall [12] asked for the minimum order $n(d, g)$ of any (d, g)-digraph. A (d, g)-digraph of order $n(d, g)$ is called (d, g)*-dicage*. Clearly, a circulant digraph $\overrightarrow{C}_n(1, 2, \ldots, d)$, where $n = (g - 1)d + 1$, is a (d, g)-digraph. Using this digraph, in [12], it was proved that $n(d, g) \leq (g - 1)d + 1$, and they proposed the conjecture $n(d, g) = d(g - 1) + 1$, that is, the order of a (d, g)-cage is at least $d(g - 1) + 1$. Caccetta and Häggkvist [13] proposed a generalization of this conjecture requiring merely a lower bound on the out-degrees of the digraph G.

Conjecture 1 *[13] Let D be a digraph on n vertices in which each vertex is of out-degree at least $d \geq 1$. Then, the girth of D is at most n/d.*

Both conjectures have been proved to be true for $d = 2$ by Behzad [14], for $d = 3$ first by Bermond and later by Hamidoune [15, 16], for $d = 4$ and for vertex-transitive digraphs by Hamidoune [17, 18]. Now, we prove Conjecture 1 in certain families of digraphs.

Corollary 5 *Let D be a digraph on n vertices, girth $g \geq 4$, minimum out-degree $\delta^+ \geq 1$ that contains a subdigraph isomorphic to an acyclic tournament of order k. Then, $g \leq n/\delta^+$, if $k \geq (\delta^+ - 1)n/\delta^+ - (\sqrt{7} - 1)\delta^+ + 4$.*

Proof It is a direct consequence of Corollary 4. □

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Data Availability This study did not use or generate any new data.

Declarations

Conflict of interest The authors declare that they have no conflict of interest in the work submitted for publication.

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