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A R Méndez<sup>1,4</sup> , A L García-Perciante<sup>1</sup>, D M Ruiz-Moreno<sup>2</sup> and A Sandoval-Villabazo<sup>2</sup><sup>1</sup> Depto. de Matemáticas Aplicadas y Sistemas, Universidad Autónoma Metropolitana-Cuajimalpa, Prol. Vasco de Quiroga 4871, Ciudad de México, 05348, México<sup>2</sup> Depto. de Física y Matemáticas, Universidad Iberoamericana, Prolongación Paseo de la Reforma 880, Ciudad de México 01219, México<sup>4</sup> Author to whom any correspondence should be addressedE-mail: [amendez@correo.cua.uam.mx](mailto:amendez@correo.cua.uam.mx), [algarcia@correo.cua.uam.mx](mailto:algarcia@correo.cua.uam.mx) and [alfredo.sandoval@ibero.mx](mailto:alfredo.sandoval@ibero.mx)**Keywords:** Navier–Stokes–Fourier system, gravitational effects, discontinuity present**Abstract**

A detailed analysis of the dynamics of unstable modes present in the linearized Navier–Stokes–Fourier system in the presence of a gravitational field is carried out. The transition between the non-dissipative and dissipative regimes is explored from the mathematical point of view by considering the same equation of state in both cases, a procedure which sheds light on the discontinuity present in the critical parameter. It is also shown that two distinct behaviours for the propagation of unstable modes can be identified below the critical value for a gravitational instability to occur, depending on a parameter  $R$  that quantifies the relevance of dissipation relative to gravitational effects.

**1. Introduction**

Structure formation is a topic of great interest, which has been addressed by several authors in the last decades [1–7]. In very general terms, the model for such a process consists in considering a homogeneous gas in which a spontaneous fluctuation in density locally gives rise to a gravitational field. As the density fluctuations increase, the balance between gas pressure and gravitational potential gradient can be broken for fluctuations with a wave number below a critical value, namely the Jeans wave number [8, 9]. When the corresponding criterion is met, a gravitational collapse is initiated, leading to structure formation.

In his original work, Jeans considered an ideal non-dissipative system for which the linearized transport equations for fluctuations can be written as

$$\frac{\partial(\delta n)}{\partial t} + n_0 \delta \theta = 0, \quad (1)$$

$$\frac{\partial(\delta \theta)}{\partial t} + \frac{1}{mn_0} \nabla^2(\delta p) = -4\pi G m \delta n, \quad (2)$$

$$C_n n_0 \frac{\partial(\delta T)}{\partial t} + p_0 \delta \theta = 0, \quad (3)$$

where  $\theta$  is the divergence of the hydrodynamic velocity,  $n$  the local particle density,  $m$  the mass of the molecules,  $p$  the hydrostatic pressure,  $T$  the temperature,  $G$  the gravitational constant and  $C_n$  the specific heat at constant density. The state variables are considered to have a constant value and a fluctuating component as follows

$$X = X_0 + \delta X, \quad (4)$$

The stability of the system of equations above can be analyzed by direct algebraic manipulation in Fourier-Laplace space in order to yield the following dispersion relation:

$$s^2 = 1 - \frac{q^2}{q_J^2}. \quad (5)$$

In equation (5),  $s$  and  $q$  are the corresponding Fourier and Laplace transform parameters, and  $q_J^2 = 4\pi Gmn_0/C_s^2$  is the Jeans wavenumber with  $C_s$  being the adiabatic speed of sound. The relation given by equation (5) implies that the system is stable for  $q^2 > q_J^2$ . Such criterion has been widely employed to roughly determine the mass required in a molecular cloud for a gravitational collapse to be feasible.

Even though the criterion described above serves as a good approximation, other factors may in principle enhance or hinder the growth of fluctuations, such as dissipation and external fields. The analysis of the effect that such phenomena have in the Jeans problem is not straightforward and has been studied to certain extent [1–7]. Indeed, when dissipation is taken into account, the dispersion relation leads to a complete cubic polynomial and thus a more involved analysis is required in order to establish the threshold for exponential growth of fluctuations as well as to explore their dynamics. Several attempts have been made in order to account for thermal and viscous dissipation in the onset of the collapse using different approaches. Kumar showed, applying the Routh-Hurwitz criterion, that viscous dissipation, rotation and the presence of a magnetic field do not affect the Jeans number in the case of an ionized gas [5]. However, thermal dissipation was shown to alter the criterion in a factor, which does not depend on thermal conductivity, as follows

$$q_{JD}^2 = \gamma \frac{4\pi Gmn_0}{C_s^2}, \quad (6)$$

where  $\gamma$  is the adiabatic index [3, 6]. Clearly, this jump in the parameter represents an abrupt change in the behavior of the solutions to the system and does not reduce exactly to the non-dissipative value when the corresponding transport coefficient is taken to be zero. On the other hand, alternative methods have also been used in order to approximate the solutions to the dispersion relation and to study the corresponding threshold and dynamics of fluctuations. In particular, in [3, 4] a factorization for the dispersion relation is proposed, however the threshold in those cases does depend on transport coefficients, opposite to the criterion obtained analytically in [5].

The aim of the present paper is to revisit the study of density fluctuation dynamics in the presence of dissipation for a self-gravitating system in a thorough and mostly analytical fashion. In particular, the transition between the Jeans wave numbers corresponding to the ideal and dissipative scenarios is analyzed. Also, two specific features of the dynamics of fluctuations below the critical parameter are studied and two regimes are identified.

In order to establish and describe these results, the rest of this work is organized as follows. In section 2, the linearized transport equations within the Navier–Stokes–Fourier regime are shown and a dimensionless dispersion relation is obtained. The Routh-Hurwitz criterion is applied in order to establish the modified criterion for the dissipative gas. Two distinct behaviors for unstable modes are identified below the critical value in section 3. Section 4 is devoted to the analysis of the scenario which includes the low dissipation case and the transition to the result obtained without dissipation is addressed. A thorough discussion of the results and final remarks are included in section 5.

## 2. Transport equations and Routh-Hurwitz stability analysis

For a self-gravitating system in the linearized Navier–Stokes–Fourier regime, the set of hydrodynamic equations read

$$\frac{\partial(\delta n)}{\partial t} + n_0 \delta \theta = 0, \quad (7)$$

$$\frac{\partial(\delta \theta)}{\partial t} + \frac{p_0}{\rho_0} \left( \frac{\nabla^2(\delta T)}{T_0} + \frac{\nabla^2(\delta n)}{n_0} \right) - \frac{4}{3} \frac{\eta}{mn_0} \nabla^2(\delta \theta) = -4\pi Gm\delta n, \quad (8)$$

$$C_n n_0 \frac{\partial(\delta T)}{\partial t} + p_0 \delta \theta - \frac{\kappa}{T_0} \nabla^2(\delta T) = 0, \quad (9)$$

where  $\eta$  is the shear viscosity and  $\kappa$  the thermal conductivity. For the sake of simplicity, a monoatomic ideal gas is here considered and the transport coefficients are expressed in terms of a relaxation time  $\tau$ , following the hatnagar, Gross and Krook (BGK) method [10] in Boltzmann's equation. That is,

$$\eta = nkT\tau, \quad \kappa = \frac{5}{3}nkT\tau, \quad C_s^2 = \frac{5}{3} \frac{kT}{m}. \quad (10)$$

The set of equations (7)–(9) can be transformed into a system of algebraic equations by performing successive Fourier transforms in space, and Laplace transforms in time. The system leads to a third order polynomial dispersion relation which can be expressed as

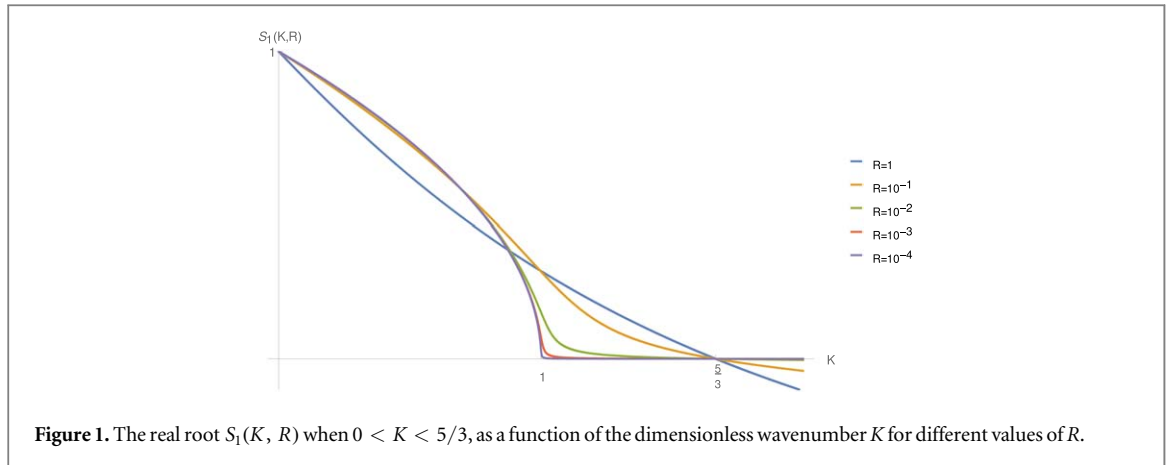


Figure 1. The real root  $S_1(K, R)$  when  $0 < K < 5/3$ , as a function of the dimensionless wavenumber  $K$  for different values of  $R$ .

$$a_3 S^3 + a_2 S^2 + a_1 S + a_0 = 0, \tag{11}$$

where the coefficients are given by

$$\begin{aligned} a_3 &= 1, \\ a_2 &= \frac{9}{5}KR, \\ a_1 &= \frac{4}{5}R^2K^2 + K - 1, \\ a_0 &= \left(\frac{3}{5}K - 1\right)RK. \end{aligned}$$

Here, the normalized wavenumber and frequency are given by  $K = (q/q_j)^2$  and  $S = s\tau_G$  respectively, where  $\tau_G = (4\pi Gmn_0)^{-1/2}$  corresponds to a gravitational characteristic time. The dissipation parameter is here defined as  $R = \tau/\tau_G$  and compares the typical collisional (microscopic) time to the characteristic gravitational timescale.

The Routh-Hurwitz criterion can be readily applied to equation (11), by means of which the number of roots on the right side quadrants of the complex plain correspond to the number of sign changes in the array  $\{a_3, a_2, b_1, a_0\}$  with

$$b_1 = \frac{a_2 a_1 - a_3 a_0}{a_2}.$$

Stability requires that all roots of equation (11) have negative real parts, such that fluctuations decay in time. Since  $a_3$  and  $a_2$  are positive, the condition for a stable system is given by

$$0 < a_0 < a_2 a_1.$$

This condition is satisfied for  $R \neq 0$ ,

$$K > \frac{5}{3}, \tag{12}$$

and

$$\frac{6}{5}R^2K^2 + K - \frac{2}{3} > 0, \tag{13}$$

simultaneously. Clearly, equation (13) is always satisfied as long as equation (12) holds true and thus the criterion for stability is given by  $K > 5/3$ , which is independent of the particular value of  $R$  provided that  $R \neq 0$ . That is, if  $R > 0$  and  $K > 5/3$  the three roots of equation (11) have negative real parts which yields an exponential decay of fluctuations in time and thus a stable system.

On the other hand for  $0 < K < 5/3$  one has  $a_0 < 0$  which indicates only one sign change in  $\{a_3, a_2, b_1, a_0\}$ . This means that one, and only one of the three roots lies in the right side of the complex plane and is thus real. Let  $S_1(K, R)$  denote such root. Figure 1 shows a plot of  $S_1(K, R)$  as a function of  $K$  for several values of  $R$  where it can be seen that this root is always real and changes sign in  $K = 5/3$ . Also notice that for  $R \rightarrow 0$ ,  $S_1$  approaches zero rapidly around  $K = 1$ , having an almost vertical tangent. However, as will be discussed in section 4,  $S_1(K, R) = 0$  only for  $K = 5/3$  if  $R \neq 0$ . The nature of the remaining roots depends strongly on the value of the dissipation parameter  $R$  and is analyzed in the next section.

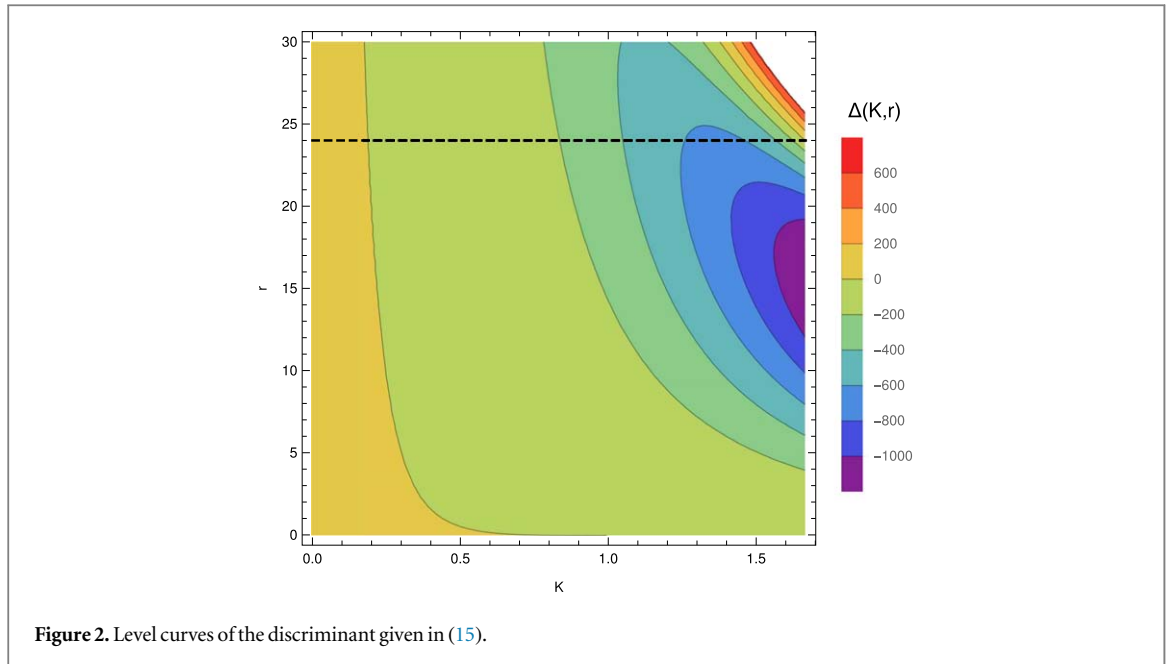


Figure 2. Level curves of the discriminant given in (15).

### 3. Dynamics of unstable modes

One of the three roots of the dispersion relation is real for all values of  $K$ , independently of the parameter  $R$  (as long as  $R > 0$ ). In this section, the behavior of the two remaining roots, which we call  $S_2$  and  $S_3$ , is analyzed within the unstable region in Laplace space, i. e.  $0 \leq K < 5/3$ . It has already been shown, based on the Routh-Hurwitz criterion, that the real part of  $S_2$  and  $S_3$  is negative, corresponding to stable modes. However, the details of the propagation, that is whether fluctuations oscillate or are purely damped, depends on the complex or real nature of these roots. This is determined by the sign of the discriminant of the dispersion relation

$$d = a_1^2 a_2^2 - 4a_0 a_2^3 - 4a_1^3 a_3 + 18a_0 a_1 a_2 a_3 - 27a_0^2 a_3^2, \tag{14}$$

which is a sixth order polynomial both in  $K$  and  $R$  and thus its analysis is not straightforward. However, since it features only even powers of  $R$ , we define  $r = R^2$  in order to simplify the notation. The condition for the presence of complex roots is written as  $\Delta(K, r) < 0$  where  $\Delta$  is defined as

$$\Delta(K, r) = \frac{4}{625}(4r^3 K^6 - 147r^2 K^5 + (525r - 15r^2)K^4 - (1050r + 625)K^3 - (150r - 1875)K^2 - 1875K + 625). \tag{15}$$

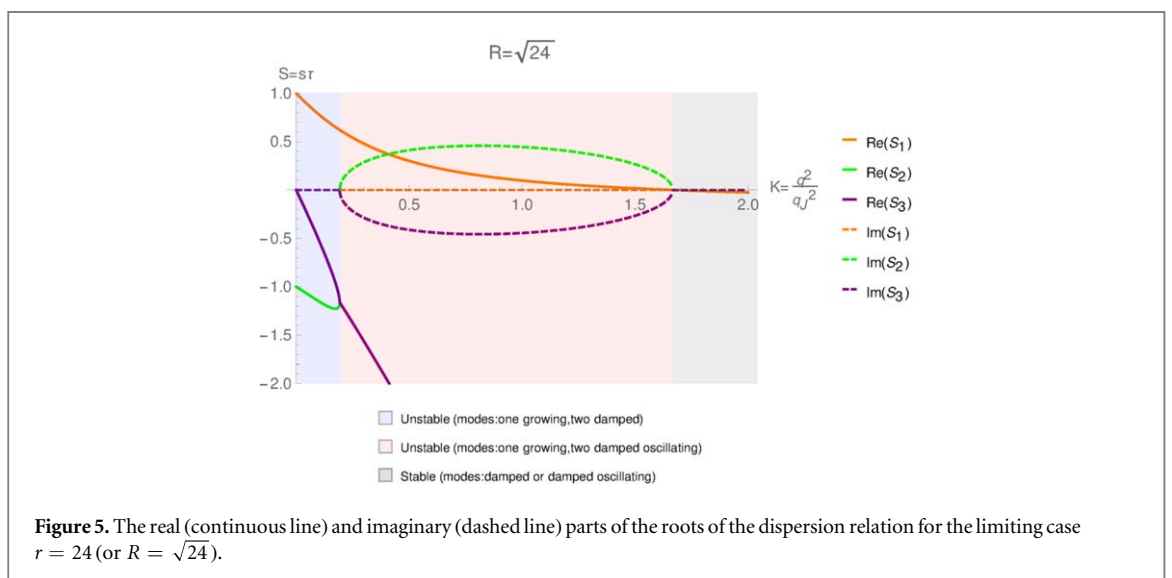
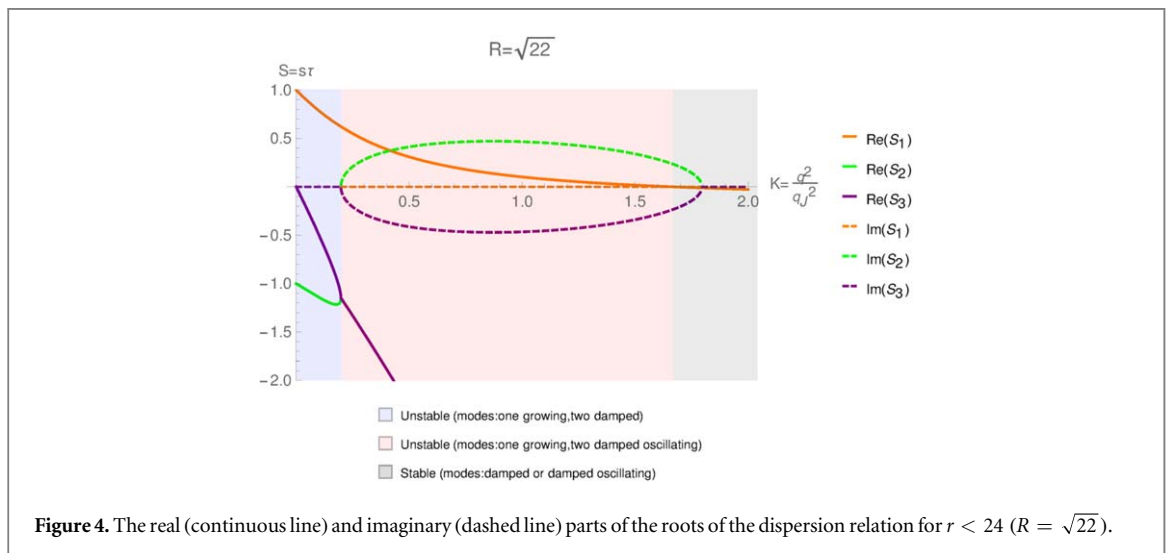
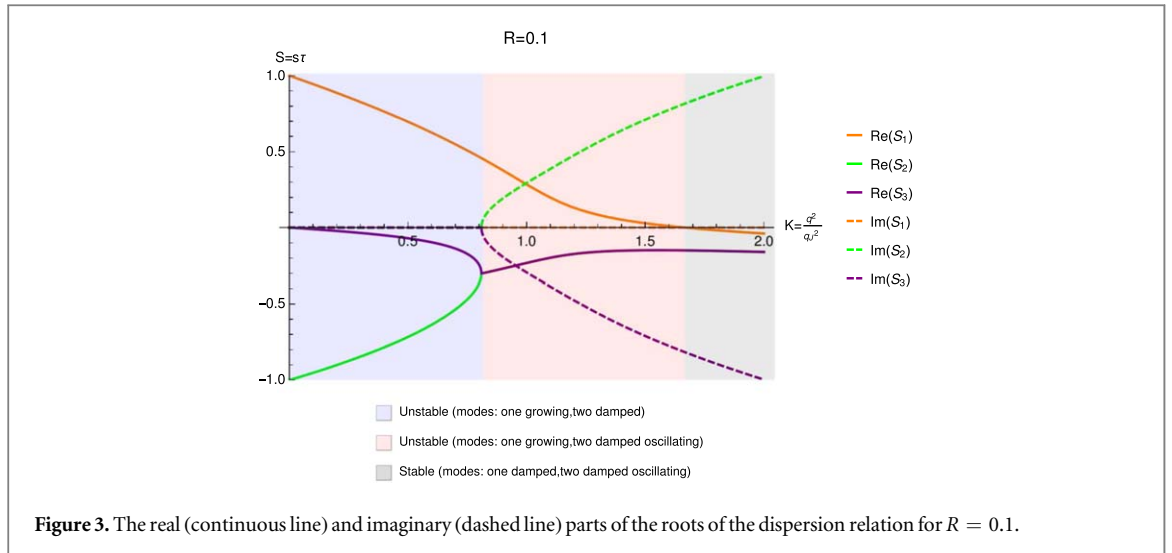
Since  $\Delta(0, r) > 0$ , one has three real roots with single multiplicity when  $K = 0$ . However,  $S_2$  and  $S_3$  may become complex if  $\Delta(K, r)$  becomes negative at some  $K$ , for a particular value of  $r$ . Since

$$\Delta\left(\frac{5}{3}, r\right) = \left(\frac{2}{27}(3 + 10r)\right)^2 (r - 24), \tag{16}$$

it is clear that for  $r < 24$  the discriminant changes sign at least once within the unstable region. Figure 2 shows that for  $(K, r) \in [0, 5/3] \times [0, 24]$ ,  $\Delta$  changes sign only once.

The change of sign in the discriminant indicates that there is a critical wave number for which the stable modes become complex and thus would lead to a doublet in a scattering experiment spectrum [11]. Moreover, once these modes change in nature, their damping weakens and thus the characteristic damping time increases. Figure 3 shows the different behaviors of the three modes for  $R = 0.1$ . This behavior corresponds to a low dissipation case, in which the characteristic form of a Rayleigh-Brillouin spectrum is obtained for  $K > 5/3$ , and the Brillouin doublet is present even within the unstable region.

Figure 4 shows that there is a second root, and change of sign in  $\Delta$ , into the stable region which approaches the critical value  $K = 5/3$  as  $r \rightarrow 24$ . The  $r = 24$  limiting case is illustrated in figure 5 where the change in behavior of the conjugate roots occurs simultaneously with the onset of stability. In that case one expects to obtain a finite spectrum for  $K > 5/3$  featuring only a superposition of three central peaks with no Brillouin doublet. For  $r > 24$  (see figure 6) the doublet also disappears within the unstable region and could not be observed for wavenumbers above  $5/3$ .



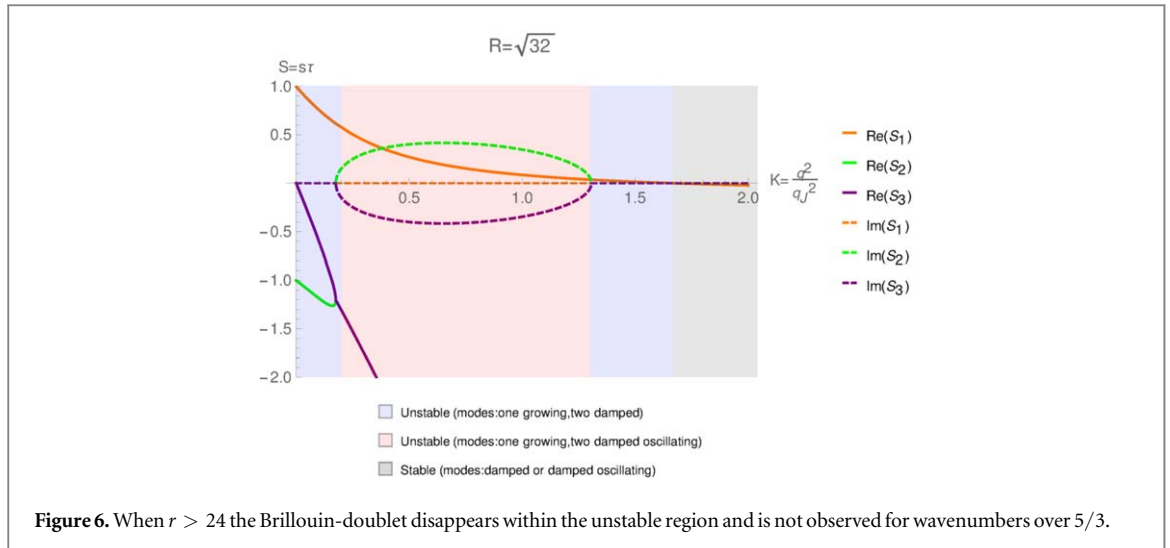


Figure 6. When  $r > 24$  the Brillouin-doublet disappears within the unstable region and is not observed for wavenumbers over  $5/3$ .

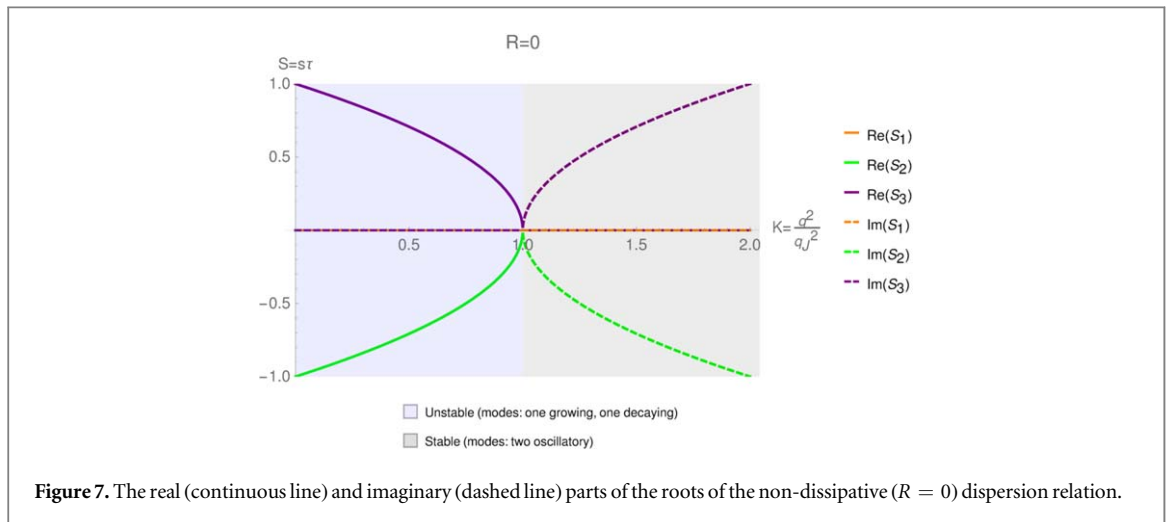


Figure 7. The real (continuous line) and imaginary (dashed line) parts of the roots of the non-dissipative ( $R = 0$ ) dispersion relation.

#### 4. Asymptotic behavior for low dissipation

The analysis carried out in the previous sections leads naturally to the question on whether the non-dissipative limit is obtained when  $R \rightarrow 0$ . That is, as strongly emphasized in section 1, the critical wavenumber for structure formation is given by  $K = 5/3$  for  $R \neq 0$ . However, if  $R = 0$  one finds the reported value in the literature corresponds to  $K = 1$  [12]. Moreover, in the non-dissipative case the dispersion relation is a quadratic polynomial which has two real roots for  $K < 1$  and two imaginary roots if  $K > 1$  (see figure 7).

Figures 7–9 show the roots of the discriminant for  $R = 0$  and  $R \rightarrow 0$  respectively. Qualitatively, the behavior is similar, however, since  $\Delta(K, R) \sim -4(K - 1)^3 + \mathcal{O}(R^2)$  the value for which the discriminant changes sign approaches one but the real part of  $S_1$  still only changes sign at  $K = 5/3$ . The similarity in the behavior observed graphically corresponds to the fact that the real part of  $S_1$  becomes very close to zero abruptly near  $K = 1$  but remains positive until the critical value  $5/3$  is attained.

#### 5. Summary and concluding remarks

In this work, a detailed mathematical analysis of the dynamics of density fluctuations in the linear regime for a monoatomic ideal gas in the presence of dissipation has been carried out. The gravitational potential generated by such fluctuations can lead to structure formation for wave numbers below a critical value.

The analysis consisted of a detailed study of the dispersion relation arising from the linearized set of hydrodynamic equations: mass, momentum and internal energy balance. The critical wavenumber has been found to differ from the usual Jean’s wavenumber by a factor of  $5/3$ , independently of the value of the transport coefficients as long as they don’t identically vanish and an internal energy balance equation, featuring dissipative effects however negligible they become, is retained.

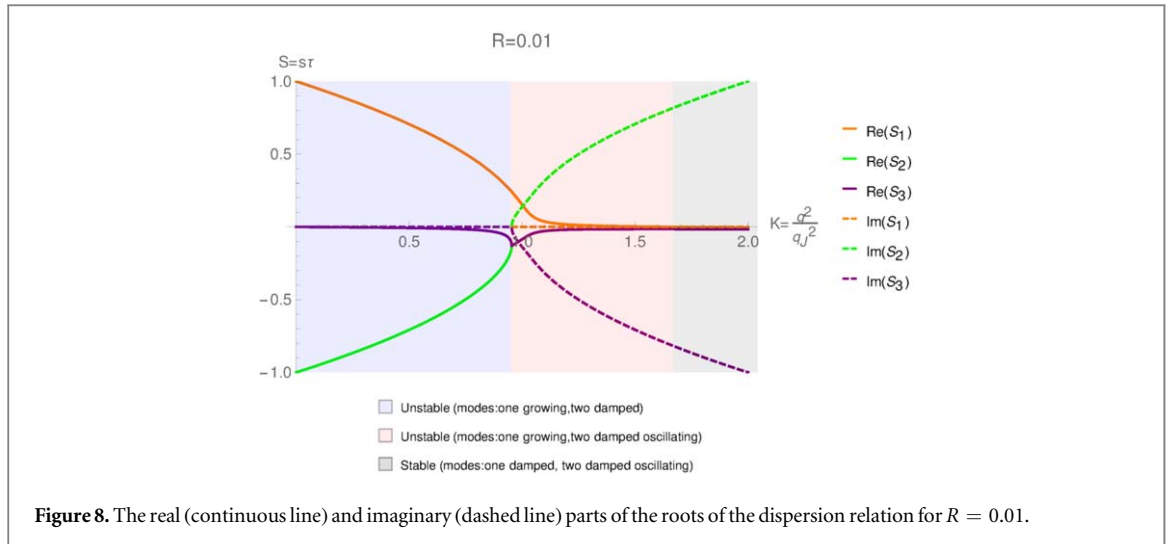


Figure 8. The real (continuous line) and imaginary (dashed line) parts of the roots of the dispersion relation for  $R = 0.01$ .

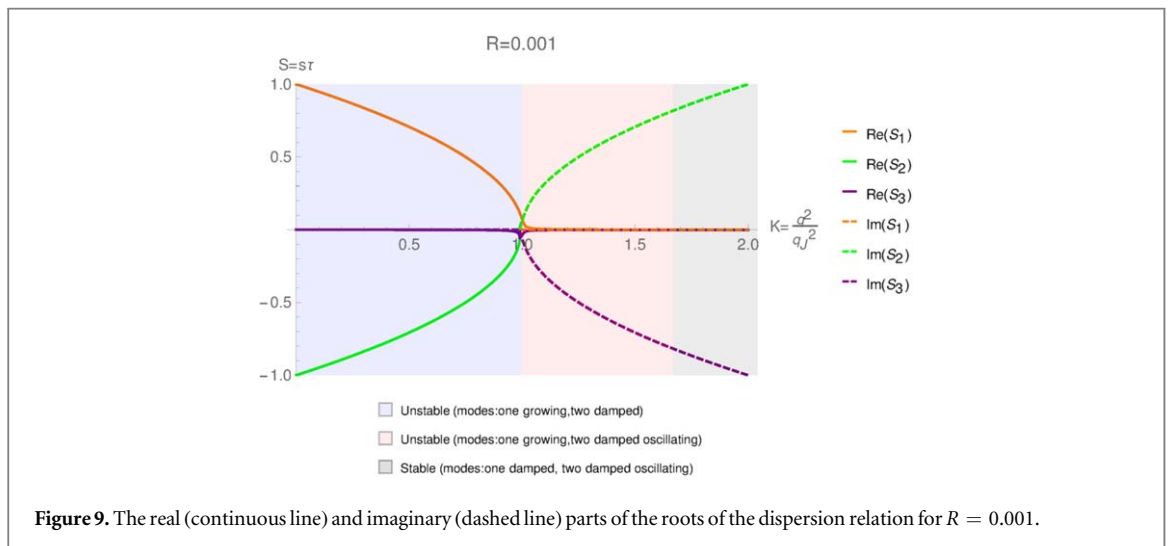


Figure 9. The real (continuous line) and imaginary (dashed line) parts of the roots of the dispersion relation for  $R = 0.001$ .

Transitions in the behavior of the stable modes have been identified, depending on the value of the ratio between the characteristic times involved in the problem: gravitational and microscopic (collisional). When dissipative effects are non-negligible, the stable modes within the stable region are purely damped and this behavior can change to damped oscillations, with smaller damping rates, either within the unstable region or in the stable range of wave numbers. This is relevant since real roots of the dispersion relation lead to central narrow peaks while evidence of complex values comes in form of a doublet of broader peaks. This information could in principle be verified in a light scattering experiment [11].

The calculations here presented, shed some light on the question on whether the critical wave number corresponds to the usual Jeans wave number or the slighter larger value obtained considering dissipation. Indeed, the strict minimum wavelength that can lead to structure formation is given by  $2\pi(5q_j/3)^{-1}$ . However, in the limit of weak dissipation, the growth rate of the instability becomes negligible when the fluctuations' wavelengths exceed  $2\pi q_j^{-1}$ . It is important to emphasize that although fluctuations do not grow significantly for  $1 \leq q/q_j < 5/3$ , they become damped only for  $q > 5q_j/3$ . These elements allow for a definitive description of the non-dissipative instability mechanism as a limit of the dissipative one without the need for the consideration of distinct equations of state depending on whether an internal energy balance equation is considered in the set of hydrodynamic equations.

The problem here addressed assumes the gravitational field is weak enough such that the classical approach is adequate. A relativistic formalism, including a fluctuating metric can be found in [8] in which corrections due to dissipation, metric fluctuations and high temperature are taken into account. A relativistic version of the calculations presented in the present paper might be helpful in the study of the contribution of each of the three effects mentioned for the hot, general relativistic, dissipative ideal gas. Such a formalism will be developed in the near future.



## ORCID iDs

A R Méndez  <https://orcid.org/0000-0001-7870-5236>

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