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MULTIPLE SOLUTIONS FOR A SINGULAR SEMILINEAR ELLIPTIC PROBLEMS WITH CRITICAL EXPONENT AND SYMMETRIES

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ABSTRACT . We consider the singular semilinear elliptic equation $-\Delta u - \frac{\mu}{|x|^2}u - \lambda u = f(x) \mid u \mid^{2^*-1}$ in $\Omega, u = 0$ on $\partial\Omega$, where Ω is a smooth bounded domain , in $\mathbb{R}^N, N \geq 4, 2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent , $f : \mathbb{R}^N \to \mathbb{R}$ is a continuous function , $0 < \lambda < \lambda_1$, where λ_1 is the first Dirichlet eigenvalue of $-\Delta - \frac{\mu}{|x|^2}$ in Ω and $0 < \mu < \bar{\mu} := (\frac{N-2}{2})^2$. We show that if Ω and f are invariant under a subgroup of $O(N)$, the effect of the equivariant topology of Ω will give many symmetric nodal solutions , which extends previous results of Guo and Niu [8] .

1 . Introduction

Much attention has been paid to the singular semilinear elliptic problem

$$-\Delta u - \mu\frac{u}{\mid x \mid^2} - \lambda u = f(x) \mid u \mid^{2^*-2} u \quad \text{in}\Omega, \tag{1.1}$$

$$u = 0 \quad \text{on}\partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 4)$ is a smooth bounded domain , $0 \in \Omega$, $0 \leq \mu < \bar{\mu} := ((N-2)/2)^2, \lambda \in (0, \lambda_1)$, where λ_1 is the first Dirichlet eigenvalue of $-\Delta - \frac{\mu}{|x|^2}$ on Ω and $2^* := 2N/(N-2)$ is the critical Sobolev exponent , and f is a continuous function . We st ate some related work here about this problem .

Brezis and Nirenberg [2] proved the existence of one positive solution for (1 . 1) with $\mu = 0$ and $f = 1$, with $\lambda \in (0, \lambda_1)$, where λ_1 is the first Dirichlet eigenvalue of $-\Delta$ on Ω and $N \geq 4$. Rey [1 3] and Lazzo [1 1] established a close relationship between the number of positive solutions for (1 . 1) with $\mu = 0$ and $f = 1$ and the domain topology if λ is positive and sufficiently small . Cerami , Solimini , and Struwe [6] proved that (1 . 1) with $\mu = 0$ and $f = 1$ has one solution changing sign exactly once for $N \geq 6$ and $\lambda \in (0, \lambda_1)$. In [5] Castro and Clapp proved that there is an effect of the domain topology on the number of minimal nodal solutions changing

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sign just once of (1 . 1) with $\mu = 0$ and $f = 1$, with λ positive sufficiently small . Recently Cano and Clapp [3] proved the multiplicity of sign changing solutions for (1 . 1) with $\lambda = a$ and $\mu = 0$, where a and f are continuous functions . The existence of non trivial positive solution for (1 . 1) with $f = 1$ and $\mu \in [0, \text{-}\mu - 1]$ and $\lambda \in (0, \lambda_1)$ where λ_1 is the first Dirichlet eigenvalue of $-\Delta - line - mu|_{x|}2$ on Ω, was proved
by Janelli [1 0] . Cao and Peng [4] proved the existence of a pair of sign changing solutions for (1 . 1) with $f = 1, N \geq 7, \mu \in [0, \text{-}\mu - 4], \lambda \in (0, \lambda_1)$. Han and Liu [9] proved the existence of one non trivial solution for (1 . 1) with $\lambda > 0, f(x) > 0$ and some additional assumptions . Chen [7] proved the existence of one positive solution for (1 . 1) with $\lambda \in (0, \lambda_1)$ and f not necessarily positive but satisfying additional hypothesis . Guo and Niu [8] proved the existence of a symmetric nodal solution and a positive solution for $0 < \lambda < \lambda_1$, where λ_1 is the first Dirichlet eigenvalue of $-\Delta - \frac{\mu}{|x|^2}$ on Ω, with Ω and f invariant under a subgroup of $O(N)$.

2 . Statement of results

Let Γ be a closed subgroup of the orthogonal transformations $O(N)$. We consider the problem

$$-\Delta u - \mu \frac{u}{| x |^2} - u\lambda \overset{u}{=} 0 =_{\text{on}} f(x)\partial_{\Omega}^{|u|^{2^*-2}u} \quad \text{in}\Omega \tag{2.1}$$
$$u(\gamma x) = u(x) \quad \forall x \in \Omega, \gamma \in \Gamma,$$

where Ω is a smooth bounded domain , $\Gamma-$ invariant in $\mathbb{R}^N, N \geq 4, 2^* := (2N)/(N-2)$ is the critical Sobolev exponent , $f : \mathbb{R}^N \to \mathbb{R}$ is a $\Gamma-$ invariant continuous function , $0 < \lambda < \lambda_1$, where λ_1 is the first Dirichlet eigenvalue of $-\Delta - \frac{\mu}{|x|^2}$ on Ω and

$$0 < \mu < \text{-}\mu := ((N-2)/2)^2.$$

Note that a subset X of $\mathbb{R}^N$ is $\Gamma-$ invariant if $\gamma x \in X$ for all $x \in X$ and $\gamma \in \Gamma$. A function $h : X \to \mathbb{R}$ is $\Gamma-$ invariant if $h(\gamma x) = h(x)$ for all $x \in X$ and $\gamma \in \Gamma$. Let $\Gamma x := \{\gamma x : \gamma \in \Gamma\}$ be the $\Gamma-$ orbit of a point $x \in \mathbb{R}^N$, and $\#\Gamma x$ its cardinality .
Let $X/\Gamma := \{\Gamma x : x \in X\}$ denote the $\Gamma-$ orbit space of $X \subset \mathbb{R}^N$ with the quotient topology .

Let us recall that the least energy solutions of

$$-\Delta_u u_{\to} =| 0u |^2_{\text{as}} {}^{*-}\frac{2_u}{|x|}\text{in}^{\mathbb{R}^N}_{\to\infty} \tag{2.2}$$

are the instantons

$$U_0^{\varepsilon,y}(x) := C(N)(\frac{\varepsilon}{\varepsilon^2 + | x - y |^2})(N-2)/2, \tag{2.3}$$

where $C(N) = (N(N-2))^{(N-2)/2}$(see [1] , [1 5]) . If the domain is not $\mathbb{R}^N$, there is no minimal energy solutions . These solutions minimize

$$S_0 := u \in D^1 \min_{,2(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} | \nabla u |^2 \, dx}{(\int_{\mathbb{R}^N} | u |^{2*} \, dx)^{2/2*}},$$

where $D^{1,2}(\mathbb{R}^N)$ is the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\| u \| 2 := \int_{\mathbb{R}^N} | \nabla u |^2 \, dx.$$

 Also, for $0 < \mu < \bar{\mu}$ it is well known that the positive solutions to

$$-\Delta u - \mu \frac{u}{|x|^2} = |u|^{2^*-2} u \quad \text{in} \mathbb{R}^N \tag{2.4}$$
$$u \to 0 \quad \text{as } |x| \to \infty.$$

are

$$U_\mu(x) := C_\mu(N)\Big(\frac{\varepsilon}{\varepsilon^2 |x|^{(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})/\sqrt{\bar\mu}} + |x|^{(\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu})/\sqrt{\bar\mu}}}\Big)^{(N-2)/2},$$

where $\varepsilon > 0$ and $C_\mu(N) = \big(\frac{4N(\bar\mu-\mu)}{N-2}\big)^{(N-2)/4}$ (see [16]). These solutions minimize

$$S_\mu := \min_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u|^2 - \mu \frac{u^2}{|x|^2}) dx}{(\int_{\mathbb{R}^N} |u|^{2^*} dx)^{2/2^*}}.$$

We denote

$$M := \{y \in \bar\Omega : \frac{\#\Gamma y}{f(y)^{(N-2)/2}} = \min_{x\in\bar\Omega} \frac{\#\Gamma x}{f(x)^{(N-2)/2}}\}.$$

We shall assume that f satisfies: (F1) $f(x) > 0$ for all $x \in \bar\Omega$.

(F2) f is *locally flat* at M, that is, there exist $r > 0, \nu > N$ and $A > 0$ such that

$$|f(x) - f(y)| \le A |x - y|^\nu \quad \text{if} y \in M \text{and } |x - y| < r.$$

For all $0 < \mu < \bar\mu$ and $0 < \lambda < \lambda_1$ we define the bilinear operator $\langle\cdot,\cdot\rangle_{\lambda,\mu}$:

$$H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R} \text{by}$$
$$\langle u, v\rangle_{\lambda,\mu} := \int_\Omega (\nabla u \cdot \nabla v - \mu \frac{uv}{|x|^2} - \lambda uv) dx$$

which is an inner product in $H_0^1(\Omega)$. Its induced norm

$$\| u \|_{\lambda,\mu} := \sqrt{\langle u, u\rangle_{\lambda,\mu}} = \Big(\int_\Omega (|\nabla u|^2 - \mu \frac{u^2}{|x|^2} - \lambda |u|^2) dx\Big)^{1/2}$$

is equivalent to the usual norm $\| u \| := \| u \|_{0,0}$ in $H_0^1(\Omega)$. This fact is a direct consequence of the Hardy inequality

$$\int_\Omega \frac{u^2}{|x|^2} dx \le \frac{1}{\bar\mu} \int_\Omega |\nabla u|^2 dx, \quad \forall u \in H_0^1(\Omega). \tag{2.5}$$

Since λ_1 is the first Dirichlet eigenvalue of $-\Delta - \frac{\mu}{|x|^2}$ on Ω,

$$\int_\Omega \lambda |u|^2 dx \le \frac{\lambda}{\lambda_1} \int_\Omega \Big(|\nabla u|^2 - \mu \frac{u^2}{|x|^2}\Big) dx. \tag{2.6}$$

Therefore, by (2.5),

$$\| u \| 2\lambda, \mu := \int_\Omega (| \nabla u |^2 - \mu \frac{u^2}{| x |^2} - \lambda \mid u |^2) dx$$

$$\geq \geq (1 - - \frac{\frac{\lambda}{\lambda_{1\lambda}}}{\lambda_1}) (\int_\Omega (1 - \frac{\mu |}{\mu}) \nabla u |^2 \int_\Omega - | \nabla \mu \frac{u^2}{|x|^2}) u|^2 dx dx, \tag{2.7}$$

$$= (1 - \frac{\lambda}{\lambda_1})(1 - \frac{\mu}{\bar{\mu}}) \| u \|^2 .$$

The other inequality follows from the Sobolev imbedding theorem .

It is easy to see that, if $f \in C(\overline{\Omega})$ satisfies (F1) then the norms

$$| u |_{2^*} := (\int_\Omega | u |^{2^*} dx)^{1/2^*}, \quad \text{and} \quad | u |_{f,2^*} := (\int_\Omega f(x) | u |^{2^*} dx)^{1/2^*}$$

are equivalent. We denote

$$\ell_f^\Gamma := \Big(\min_{x\in\Omega} \frac{\#\Gamma x}{f(x)^{(N-2)/2}}\Big) S_0^{N/2}.$$

Our multiplicity results will require the following non existence assumption. (A1) The problem

$$\begin{gathered} -\Delta u = f(x) | u |^{2^*-2} u \quad \text{in}\Omega \\ u = 0 \quad \text{on}\partial\Omega \\ u(\gamma x) = u(x) \quad \forall x \in \Omega, \gamma \in \Gamma \end{gathered} \tag{2.8}$$

does not have a positive solution u which satisfies $\| u \|^2 \leq \ell_f^\Gamma$.

2.1. Multiplicity of positive solutions.

Our next result generalizes the work of Guo and Niu [8] for the problem (2.1) and establishes a relationship between the topology of the domain and the multiplicity of positive solutions. For $\delta > 0$ let

$$M_\delta^- := \{y \in M : \mathrm{dist}(y, \partial\Omega) \geq \delta\}, B_\delta(M) := \{z \in \mathbb{R}^N : \mathrm{dist}(z, M) \leq \delta\}. \tag{2.9}$$

Theorem 2.1. *Let $N \geq 4$, Ω and f be $\Gamma-$ invariant, and (F1), (F2), (A1) and $\ell_f^\Gamma \leq S_\mu^{N/2}$ hold. For each $\delta, \delta' > 0$ there exist $\lambda^* \in (0, \lambda_1), \mu^* \in (0, \overline{\mu})$ such that for all $\lambda \in (0, \lambda^*), \mu \in (0, \mu^*)$ the problem (2.1) has at least*

$$\mathrm{cat}_{B_\delta(M)/\Gamma}(M_\delta^-/\Gamma)$$

positive solutions which satisfy

$$\ell_f^\Gamma - \delta' \leq \| u \|_{\lambda,\mu}^2 < \ell_f^\Gamma.$$

2.2. Multiplicity of nodal solutions.

We assume that Γ is the kernel of an epimorphism $\tau : G \to \mathbb{Z}/2 := \{-1, 1\}$, where G is a closed subgroup of $O(N)$ for which, Ω is $G-$ invariant and $f : \mathbb{R}^N \to \mathbb{R}$ is a $G-$ invariant function.

A real valued function u defined in Ω will be called $\tau-$ equivariant if

$$u(gx) = \tau(g)u(x) \quad \forall x \in \Omega, g \in G.$$

In this section we study the problem

$$\begin{gathered} -\Delta u - \mu\frac{u}{| x |^2} - \lambda u = f(x)|u|^{2^*-2}u \quad \text{in}\Omega \\ u = 0 \quad \text{on}\partial\Omega \\ u(gx) = \tau(g)u(x) \quad \forall x \in \Omega, g \in G \end{gathered} \tag{2.10}$$

Note that all $\tau-$ equivariant functions u are $\Gamma-$ invariant; i.e., $u(gx) = u(x)$ for all $x \in \Omega, g \in \Gamma$. If u is a $\tau-$ equivariant function then $u(gx) = -u(x)$ for all $x \in \Omega$ and $g \in \tau^{-1}(-1)$. Thus all non trivial $\tau-$ equivariant solution of (2.1) change sign.

Definition 2 . 2 . We call a $\Gamma-$ invariant subset X of $\mathbb{R}^N \Gamma-$ connected if cannot be written as the union of two disj oint open $\Gamma-$ invariant subsets . A real valued function $u : \Omega \to \mathbb{R}$ is $(\Gamma, 2)-$ nodal if the sets

$$\{x \in \Omega : u(x) > 0\} \quad \text{and} \quad \{x \in \Omega : u(x) < 0\}$$

are nonempty and $\Gamma-$ connected .

 For each $G-$ invariant subset X of $\mathbb{R}^N$, we define

$$X^\tau := \{x \in X : Gx = \Gamma x\}.$$

Let $\delta > 0$, define

$$M^-_{\tau,\delta} := \{y \in M : \operatorname{dist}(y, \partial\Omega \cap \Omega^\tau) \geq \delta\},$$

and $B_\delta(M)$ as in (2 . 9) .

The next theorem is a multiplicity result for $\tau-$ equivariant $(\Gamma, 2)-$ nodal solutions for the problem (2 . 1) .

Theorem 2 . 3 . *Let $N \geq 4$, and (F 1) , (F 2) , (A 1) and $\ell^\Gamma_f \leq S^{N/2}_\mu$ hold . If Γ is the kernel of an epimorphism $\tau : G \to \mathbb{Z}/2$ defined on a clos ed subgroup G of $O(N)$ for which Ω and f are $G-$ invariant . Given $\delta, \delta' > 0$ there exists $\lambda^* \in (0, \lambda_1), \mu^* \in (0, \overline{}_\mu)$ such that for all $\lambda \in (0, \lambda^*), \mu \in (0, \mu^*)$ the problem (2 . 1) has at least*

$$\operatorname{cat}_{(B_\delta(M)\setminus B_\delta(M)^\tau)/G}(M^-_{\tau,\delta}/G)$$

pairs $\pm u$ of $\tau-$ equivariants $(\Gamma, 2)-$ nodal s o lutions which satisfy

$$2\ell^\Gamma_f - \delta' \leq\| u \|^2_{\lambda,\mu} < 2\ell^\Gamma_f.$$

2 . 3 . **Non symmetric properties for solutions .** Let $\Gamma \subset \widetilde{\Gamma} \subset O(N)$. Next we give sufficient conditions for the existence of many solutions which are $\Gamma-$ invariant but are not $\widetilde{\Gamma}_-$ invariant .

Theorem 2 . 4 . *Let $N \geq 4$ and assume that f satisfies (F 1) , (F 2) , (A 1) and $\ell^\Gamma_f \leq S^{N/2}_\mu$. Let $\widetilde{\Gamma}$ be a c los ed subgroup of $O(N)$ containing Γ, for which Ω and f are $\widetilde{\Gamma}_{-invariant}$ and*

$$\overline{}\min_{x\in\Omega} \frac{\#\Gamma x}{f(x)^{\frac{N-2}{2}}} < \overline{}\min_{x\in\Omega} \frac{\#^{\widetilde{\Gamma}} x}{f(x)^{(N-2)/2}}.$$

Given $\delta, \delta' > 0$ th ere exist $\lambda^* \in (0, \lambda_1), \mu^* \in (0, \overline{}_\mu)$ such that for al l $\lambda \in (0, \lambda^*)$, $\mu \in (0, \mu^*)$ the problem (2 . 1) has at least

$$\operatorname{cat}_{B_\delta(M)/\Gamma}(M^-_\delta/\Gamma)$$

positive s o lutions which are not $\widetilde{\Gamma}_{-invariant}$ and satisfy

$$2\ell^\Gamma_f - \delta' \leq\| u \|^2_{\lambda,\mu} < 2\ell^\Gamma_f.$$

3 . The variational problem

Let $\tau : G \to \mathbb{Z}/2$ be a homomorphism defined on a closed subgroup G of $O(N)$, and $\Gamma := \ker \tau$. Consider the problem

$$-\Delta u - \mu\frac{u}{| x |^2} - u\lambda^u_= 0 =_{\text{on}} f(x)\partial^{|u|^{2^*-2}u}_\Omega \quad \text{in}\Omega \tag{3.1}$$
$$u(gx) = \tau(g)u(x) \quad \forall x \in \Omega, g \in G,$$

where Ω is a $G-$ invariant bounded smooth subset of $\mathbb{R}^N$, and $f : \mathbb{R}^N \to \mathbb{R}$ is a $G-$ invariant continuous function which satisfies (F 1) .

If $\tau \equiv 1$ then the problems (2 . 1 0) and (2 . 1) coincide . If τ is an epimorphism then a solution of (2 . 1 0) is a solution of (2 . 1) with the additional property $u(gx) = -u(x)$
for all $x \in \Omega$ and $g \in \tau^{-1}(-1)$. So every non trivial solution of (2 . 1 0) is a sign changing solution for (2 . 1) .

 The homomorphism τ induces the action of G on $H_0^1(\Omega)$ given by

$$(gu)(x) := \tau(g)u(g^{-1}x).$$

The fixed point space of the action is given by

$$\begin{aligned} H_0^1(\Omega)^\tau &:= \{u \in H_0^1(\Omega) : gu = u \quad \forall g \in G\} \\ &= \{u \in H_0^1(\Omega) : u(gx) = \tau(g)u(x) \quad \forall g \in G, \quad \forall x \in \Omega\}, \end{aligned}$$

is the space of $\tau-$ equivariant functions . The fixed point space of the restriction of this action to Γ

$$H_0^1(\Omega)^\Gamma = \{u \in H_0^1(\Omega) : u(gx) = \tau(g)u(x), \forall g \in \Gamma, \forall x \in \Omega\}$$

are the $\Gamma-$ invariant functions of $H_0^1(\Omega)$. The norms $\| \cdot \| \lambda, \mu$, $\| \cdot \|$ on $H_0^1(\Omega)$ and
$| \cdot | 2*$, $| \cdot |_f,$ $2*$ on $L^{2^*}(\Omega)$ are $G-$ invariant with respect to the action induced by τ; therefore , the functional

$$\begin{aligned} E_{\lambda,\mu,f}(u) &:= \frac{1}{2}\int_\Omega (| \nabla u |^2 - \mu \frac{u^2}{| x |^2} - \lambda | u |^2)dx - \frac{1}{2*}\int_\Omega f(x) | u |^{2^*} dx \\ &= \frac{1}{2} \| u \|_{\lambda,\mu}^2 - \frac{1}{2*} | u |_{f,2*}^{2^*} \end{aligned}$$

is $G-$ invariant , with derivative

$$DE_{\lambda,\mu,f}(u)v = \int_\Omega (\nabla u \cdot \nabla v - \mu \frac{uv}{| x |^2} - \lambda uv)dx - \int_\Omega f(x) | u |^{2^*-2} uvdx.$$

By the principle of symmetric criticality [1 2] , the critical points of its restriction to $H_0^1(\Omega)^\tau$ are the solutions of (2 . 1 0) , and all non trivial solutions lie on the Nehari manifold

$$\begin{aligned} \mathcal{N}_{\lambda,\mu,f}^\tau &:= \{u \in H_0^1(\Omega)^\tau : u \neq 0, DE_{\lambda,\mu,f}(u)u = 0\} \\ &= \{u \in H_0^1(\Omega)^\tau : u \neq 0, \| u \|_{\lambda,\mu}^2 = | u |_{f,2*}^{2^*}\}. \end{aligned}$$

which is of class C^2 and radially diffeomorphic to the unit sphere in $H_0^1(\Omega)^\tau$ by the radial projection

$$\pi_{\lambda,\mu,f} : H_0^1(\Omega)^\tau \setminus \{0\} \to \mathcal{N}_{\lambda,\mu,f}^\tau \quad \pi_{\lambda,\mu,f}(u) := (\frac{\| u \|_{\lambda,\mu}^2}{| u |_{f,2*}^{2*}}) (N-2)/4_u.$$

Therefore , the nontrivial solutions of (2 . 1 0) are precisely the critical points of the restriction$\mathcal{N}_{\lambda,\mu,f}$.Note$^{\text{of}}$that$^{E_{\lambda,\mu,f}}$ to $\mathcal{N}_{\lambda,\mu,f}^\tau$. If $\tau \equiv 1$ we write $\mathcal{N}_{\lambda,\mu,f}^\Gamma$ and if G is a trivial group

$$E_{\lambda,\mu,f}(u) = \frac{1}{N} \| u \|_{\lambda,\mu}^2 = \frac{1}{N} | u |_{f,2*}^{2^*} \quad \forall u \in \mathcal{N}_{\lambda,\mu,f}^\tau. \tag{3.2}$$

and

$$E_{\lambda,\mu,f}(\pi_{\lambda,\mu,f}(u)) = \frac{1}{N}(\frac{\| u \|_{\lambda,\mu}^2}{| u |_{f,2*}^2}) N/2 \quad \forall u \in H_0^1(\Omega)^\tau \setminus \{0\}.$$

We define

$$
\begin{aligned}
m(\lambda, \mu, f) &:= \inf_{\mathcal{N}_{\lambda,\mu,}} {}_f E_{\lambda,\mu,f}(u) = \inf_{\mathcal{N}_{\lambda,\mu,}} {}_f \frac{1}{N} \parallel u \parallel^2_{\lambda,\mu} \\
&= u \in 0^{\inf_1}_H(\Omega) \setminus \{0\} \frac{1}{N} (\frac{\parallel u \parallel^2_{\lambda,\mu}}{\mid u \mid^2_{f,2*}}^{)} N/2.
\end{aligned}
$$

 In particular , $E_{\lambda,\mu,f}$ are bounded below on $\mathcal{N}_{\lambda,\mu,f}$. We denote by

$$m^{\Gamma}(\lambda,\mu,f) := \mathcal{N}^{\Gamma}{}^{\lambda,\mu,}_{\inf}{}_{f} E_{\lambda,\mu,f}, \quad m^{\tau}(\lambda,\mu,f) := \mathcal{N}\tau^{\lambda,\mu,}_{\inf}{}_{f} E_{\lambda,\mu,f}.$$

3 . 1 . **Estimates for the infimum .**

$$m^{\Gamma}(\lambda,\mu,f) > 0.$$ **Proposition**3.1.

Proo f − period Assume that $m^{\Gamma}(\lambda,\mu,f) = 0$. Then there exist a sequence (u_n) on $\mathcal{N}^{\Gamma}_{\lambda,\mu,f}$ such that

$$E_{\lambda,\mu,f}(u_n) \to m^{\Gamma}(\lambda,\mu,f) = 0.$$

So $E_{\lambda,\mu,f}(u_n) = \frac{1}{N} \parallel u_n \parallel^2_{\lambda,\mu}$. Since $\parallel \cdot \parallel \lambda,\mu$ and $\parallel \cdot \parallel$ are equivalent norms of $H^1_0(\Omega)$ we have that $u_n \to 0$ strongly in $H^1_0(\Omega)$; but $\mathcal{N}^{\Gamma}_{\lambda,\mu,f}$ is closed in $H^1_0(\Omega)$ then $0 \in \mathcal{N}^{\Gamma}_{\lambda,\mu,f}$ which is a contradiction . □

Proposition 3 . 2 . *Let* $0 < \lambda \leq \lambda' < \lambda_1$, $0 < \mu \leq \mu' <$ -μ *and* $f : \mathbb{R}^N \to \mathbb{R}$ *a continuous function* $\Sigma-$ *invariant , such that* f *satisfies (F 1) , and* Σ *is a clos ed subgroup of* $O(N)$. *Then* $\parallel u \parallel^2_{\lambda',\mu'} \leq \parallel u \parallel^2_{\lambda,\mu}$,

$$m(\lambda',\mu',f) \leq m(\lambda,\mu,f) \quad \text{and} \quad m^{\Sigma}(\lambda',\mu',f) \leq m^{\Sigma}(\lambda,\mu,f).$$

Proof . By definition of $\parallel \cdot \parallel \lambda,\mu$ we obtain the first inequality . Let $u \in H^1_0(\Omega) \setminus \{0\}$, then

$$\begin{aligned} m(\lambda',\mu',f) &\leq E_{\lambda\prime,\mu\prime,f}(\pi_{\lambda',\mu',f}(u)) \\ &= \frac{1}{N}\Big(\frac{\parallel u \parallel^2_{\lambda',\mu'}}{\mid u \mid^2_{f,2*}}\Big)^{N/2} \\ &\leq \frac{1}{N}\Big(\frac{\parallel u \parallel^2_{\lambda,\mu}}{\mid u \mid^2_{f,2*}}\Big)^{N/2} \\ &= E_{\lambda,\mu,f}(\pi_{\lambda,\mu,f}(u)). \end{aligned}$$

From this inequality there proof follows . □ We denote by λ_1 the first Dirichlet eigenvalue of $-\Delta - line - mu|_{x|}2$ in $H^1_0(\Omega)$.

Lemma 3 . 3 . *For all* $\lambda \in (0,\lambda_1), \mu \in (0, \underline{\quad}\mu), u \in H^1_0(\Omega)^{\tau}$, *it fo l lows that*

$$E_{0,0,f}(\pi_{0,0,f}(u)) \leq \Big(\frac{\bar{\mu}}{\bar{\mu}-\mu}\Big)^{\frac{N}{2}}\Big(\frac{\lambda_1}{\lambda_1-\lambda}\Big)^{\frac{N}{2}} E_{\lambda,\mu,f}(\pi_{\lambda,\mu,f}(u)).$$

Proof . Since

$$E_{\lambda,\mu,f}(\pi_{\lambda,\mu,f}(u)) = \frac{1}{N}\Big(\frac{\parallel u \parallel 2\lambda,_{\mu}}{\mid u \mid^2_{f,2*}}\Big)^{N/2} = \frac{1}{N}\Big(\frac{\parallel u \parallel^N_{\lambda,\mu}}{\mid u \mid^N_{f,2*}}\Big),$$

and by (2 . 7)

$$(1-\frac{\mu}{\bar{\mu}})(1-\frac{\lambda}{\lambda_1}) \parallel u \parallel^2 \leq \parallel u \parallel^2_{\lambda,\mu},$$

then

$$(1-\frac{\mu}{\bar{\mu}})\frac{N}{2}(1-\frac{\lambda}{\lambda_1})\frac{N}{2}\parallel u\parallel^N\leq\parallel u\parallel^N_{\lambda,\mu}$$

$$(1-\frac{\mu}{\bar{\mu}})\frac{N}{2}(1-\frac{\lambda}{\lambda_1})\frac{N}{2}\frac{1}{N}\frac{\parallel u\parallel^N}{\mid u\mid^N_{f,2*}}\leq E_{\lambda,\mu,f}(\pi_{\lambda,\mu,f}(u))$$

$$E_{0,0,f}(\pi_{0,0,f}(u)) \le (\frac{\bar{\mu}}{\bar{\mu}-\mu})^{\frac{N}{2}}(\frac{\lambda_1}{\lambda_1-\lambda})^{\frac{N}{2}} E_{\lambda,\mu,f}(\pi_{\lambda,\mu,f}(u)),$$

which concludes the proof . □ As a immediately consequence we have the following result .

Corollary 3 . 4 .

$$m^{\tau}(0,0,f) \le (\frac{\bar{\mu}}{\bar{\mu}-\mu})^{\frac{N}{2}}(\frac{\lambda_1}{\lambda_1-\lambda})^{\frac{N}{2}} m^{\tau}(\lambda,\mu,f).$$

For the proof of the next lemma we refer the reader to [3] . **Lemma 3 . 5 .** *If* $\Omega \cap M \neq \varnothing$ *th en*

$$\text{(a)} \quad m^{\Gamma}(0,0,f) \le \frac{1}{N}\ell_f^{\Gamma}.$$

(b) *if the re exists* $y \in \Omega \cap M$ *with* $\Gamma x \neq Gy$, *then* $m^{\tau}(0,0,f) \le \frac{2}{N}\ell_f^{\Gamma}$. 3 . 2 . **A compactness result . Definition 3 . 6 .** A sequence $\{u_n\} \subset H_0^1(\Omega)$ satisfying

$$E_{\lambda,\mu,f}(u_n) \to c \quad \text{and} \quad \nabla E_{\lambda,\mu,f}(u_n) \to 0.$$

is called a Palais - Smale sequence for $E_{\lambda,\mu,f}$ at c. We say that $E_{\lambda,\mu,f}$ satisfies the Palais - Smale condition $(PS)_c$ if every Palais - Smale sequence for $E_{\lambda,\mu,f}$ at c has a convergent subsequence . If $\{u_n\} \subset H_0^1(\Omega)^{\tau}$ then $\{u_n\}$ is a $\tau-$ equivariant Palais - Smale sequence and $E_{\lambda,\mu,f}$ satisfies the $\tau-$ equivariant Palais - Smale condition , $(PS)_c^{\tau}$. If $\tau \equiv 1\{u_n\}$ is a $\Gamma-$ invariant Palais - Smale sequence and $E_{\lambda,\mu,f}$ satisfies the $\Gamma-$ invariant Palais - Smale condition $(PS)_c^{\Gamma}$.

The next theorem , proved by Guo - Niu [8] , describes the $\tau-$ equivariant Palais - Smale sequence for $E_{\lambda,\mu,f}$.

Theorem 3 . 7 . *Let* (u_n) *be a Palais - Smale in* $H_0^1(\Omega)^{\tau}$, *for* $E_{\lambda,\mu,f}$ *at* $c \ge 0$. *Then there exist a s o lution* u *of* $(2.10), m, l \in \mathbb{N}$; *a clos ed subgroup* G^i *of finite index in* G, *s equences* $\{y_n^i\} \subset \Omega, \{n_r^i\} \subset (0,+\infty)$; *a s o lutio n* $i_{\widehat{u}_0}$ *of* (2 . 2) *, for* $i = 1,...,m$; *and* $\{R_n^j\} \subset \mathbb{R}^+$, *a s o lution* $j_{\widehat{u}_\mu}$ *of* (2 . 4) *for* $j = 1,...,l$. *Such that*

$$\text{(i)} \quad G_{y_n}i \quad = G^i$$

(i i) $(n_r^i)^{-1}dist(y_n^i, \partial\Omega) \to \infty, y_n^i \to y^i$, *if* $n \to \infty$, *for* $i = 1,...,m$. (ii i) $(n_r^i)^{-1} \mid gy_n^i - g'y_n^i \mid \to \infty$, *if* $n \to \infty$, *and* $[g] \neq [g'] \in G/G^i$, *for* $i = 1,...,m$,

$$\text{(iv)} \quad i_{\widehat{u}_0}(gx) = \tau(g)^i\widehat{u}_0(x) \forall z \in \mathbb{R}^N and g \in G^i,$$

$$\text{(v)} \quad j_{\widehat{u}_\mu}(gx) = \tau(g) j_{\widehat{u}_\mu}(x) \forall z \in \mathbb{R}^N \text{ and } g \in G, R_n^j \to 0 \text{ for } j = 1,...,l$$

(vi)

$$u_n(x) = u(x) + \sum_{m}^{i=1}[g]\sum_{\in G/} G^{i^{(n_r^i)\frac{2-N}{2}f(y^i)\frac{2-N}{4}\tau(g)^i\widehat{u}_0(g^{-1}(\frac{x-gy_n^i}{n_r^i}))}}$$

$$l$$

$$+\sum_{j=1}^{(R^j} n)\frac{2-N}{2}\widehat{u}i_\mu(\frac{x}{R_n^j}) + o(1),$$

(vii)

$$E_{\lambda,\mu,f}(u_n) \to E_{\lambda,\mu,f}(u) + \sum_{i=1}^{m}(\frac{\#(G/G^i)}{f(y^i)\frac{N-2}{2}})E_{0,0,1}^{\infty}(i_{\widehat{u}_0}) + \sum_{j=1}^{l} E_{0,\mu,1}^{\infty}(j_{\widehat{u}_\mu}), as$$

$$n \to \infty$$

 Corollary 3.8. *$E_{\lambda,\mu,f}$ satisfies $(PS)^{\tau}_c$ at every*

$$c < \min\{\#(G/\Gamma)\frac{\ell_f^\Gamma}{N}, \frac{\#(G/\Gamma)}{N} S_\mu^{N/2}\}.$$

4 . The bariorbit map

We will assume the nonexistence condition (NE) The infimum of $E_{0,0,f}$ is not achieved in $\mathcal{N}^\Gamma_{0,0,f}$.

Corollary 3 . 8 and Lemma 3 . 5 imply

$$m^\Gamma(0,0,f) := \inf_{\mathcal{N}^{\Gamma}_{0,0,f}} E_{0,0,f} = \Big(\min_{x\in\Omega} \frac{\#\Gamma x}{f(x)^{(N-2)/2}}\Big) \frac{1}{N} S^{N/2}. \tag{4.1}$$

if (NE) is assumed . It is well known that (NE) holds , if $\Gamma = \{1\}$ and f is constant (see [14 , Cap . III , Teorema 1 . 2]) . Set

$$M := \{y \in \Omega : \frac{\#\Gamma y}{f(y)^{(N-2)/2}} = \min_{x\in\Omega} \frac{\#\Gamma x}{f(x)^{(N-2)/2}}\}.$$

For every $y \in \mathbb{R}^N, \gamma \in \Gamma$, the isotropy subgroups satisfy $\Gamma_{\gamma y} = \gamma \Gamma_y \gamma^{-1}$. Therefore the set of isotropy subgroups of $\Gamma-$ invariant subsets consists of complete conjugacy classes . We choose $\Gamma_i \subset \Gamma, i = 1, ..., m$, one in each conjugacy class of an isotropy subgroup of M. Set

$$V^i := \{z \in V : \gamma z = z \quad \forall \gamma \in \Gamma_i\}$$

the fixed point space of $V \subset \mathbb{R}^N$ under the action of Γ_i. Set

$$M^i := \{y \in M : \Gamma_y = \Gamma_i\},$$
$$\Gamma M^i := \{\gamma y : \gamma \in \Gamma, \quad y \in M^i\} = \{y \in M : (\Gamma_y) = (\Gamma_i)\}.$$

By definition of M it follows that f is constant on each ΓM^i. *Set*

$$f_i := f(\Gamma M^i) \in \mathbb{R}.$$

Fix $\delta_0 > 0$ such that

$$|y - \gamma y| \ge 3\delta_0 \quad \forall y \in M^i, \gamma \in \Gamma, \text{ if } \gamma y \neq y, \qquad \operatorname{dist}(\Gamma M^i, \Gamma M^j) \ge 3\delta_0 \text{ if } i \neq j, \tag{4.2}$$

and such that the isotropy subgroup of each point in $M^i_{\delta_0} := \{z \in V^i : \operatorname{dist}(z, M^i) \le \delta_0\}$ is precisely Γ_i. Define

$$W_{\varepsilon,z} := \sum_{[g]\in\Gamma/\Gamma_i} f_i^{\frac{2-N}{4}} U_{\varepsilon,gz} \quad \text{if } z \in M^i_{\delta_0},$$

where $U_{\varepsilon,y} := U_0^{\varepsilon,y}$ as in (2 . 3) . For each $\delta \in (0, \delta_0)$ define

$$M_\delta := M^1_\delta \cup \cdots \cup M^m_\delta,$$
$$B_\delta := \{(\varepsilon, z) : \varepsilon \in (0,\delta), z \in M_\delta\},$$
$$\Theta_\delta := \{\pm W_{\varepsilon,z} : (\varepsilon, z) \in B_\delta\}, \quad \Theta_0 := \Theta_{\delta_0}.$$

For the proof of next proposition see [3] .

Proposition 4.1. *Let $\delta \in (0, \delta_0)$, and assume that (NE) holds. There exists $\eta > m^{\Gamma}(0,0,f)$ with following properties: For each $u \in \mathcal{N}^{\Gamma}_{0,0,f}$ such that $E_{0,0,f}(u) \le \eta$ we have*

$$\inf_{W\in\Theta_0} \| u - W \| < \sqrt{\frac{1}{2} N m^{\Gamma}(0,0,f)},$$

and there exist precisely one $\nu \in \{-1,1\}$, one $\varepsilon \in (0,\delta_0)$ and one $\Gamma-$orbit $\Gamma z \in M_{\delta_0}$ such that

$$\| u - \nu W_{\varepsilon,z} \| = \inf_{W\in\Theta_0} \| u - W \| .$$

$$Moreover (\varepsilon, z) \in B_{\delta}.$$

4.1. Definition of the bariorbit map. Fix $\delta \in (0,\delta_0)$ and choose $\eta > m^{\Gamma}(0,0,f)$ as in Proposition 4.1. Define

$$E^{\eta}_{0,0,f} := \{u \in H^1_0(\Omega) : E_{0,0,f}(u) \le \eta\},$$
$$B_{\delta}(M) := \{z \in \mathbb{R}^N : \mathrm{dist}(z, M) \le \delta\},$$

and the space of $\Gamma-$orbits of $B_{\delta}(M)$ by $B_{\delta}(M)/\Gamma$.
From Proposition 4.1 we can define **Definition 4.2.** The bariorbit map

$$\beta^{\Gamma} : \mathcal{N}^{\Gamma}_{0,0,f} \cap E^{\eta}_{0,0,f} \to B_{\delta}(M)/\Gamma,$$

is defined by

$$\beta^{\Gamma}(u) = \Gamma y de \Longleftrightarrow^{f} \quad \| u \pm W_{\varepsilon,y} \| \quad = \min_{W\in\Theta_0} \| u - W \| .$$

This map is continuous and $\mathbb{Z}/2-$ invariant by the compactness of M_{δ}.

If Γ is the kernel of an epimorphism $\tau : G \to \mathbb{Z}/2$, choose $g\tau \in \tau^{-1}(-1)$. Let $u \in \mathcal{N}^{\tau}_{0,0,f}$ then u changes sign and $u^-(x) = -u^+(g_{\tau}^{-1}x)$. Therefore, $\| u^+ \|^2 = \| u^- \|^2$ and $| u^+ |^{2^*}_{f,2*} = | u^- |^{2^*}_{f,2*}$. So

$$u \in \mathcal{N}^{\tau}_{0,0,f} \Longrightarrow u^{\pm} \in \mathcal{N}^{\Gamma}_{0,0,f} \quad \text{and} \quad E_{0,0,f}(u) = 2E_{0,0,f}(u^{\pm}). \quad (4.3)$$

Lemma 4.3. *If $E_{0,0,f}$ does not achieve its infimum at $\mathcal{N}^{\tau}_{0,0,f}$, then*

$$m^{\tau}(0,0,f) := \mathcal{N}\tau^{0,0,}_{\inf}{}_{f} E_{0,0,f} = \Big(\min_{x\in\Omega} \frac{\#\Gamma x}{f(x)^{(N-2)/2}}\Big) \frac{2}{N} S^{N/2} = 2m^{\Gamma}(0,0,f).$$

Proof. By contradiction. Suppose that there exists $u \in \mathcal{N}^{\tau}_{0,0,f}$ such that $E_{0,0,f}(u) =$

$$m^{\tau}(0,0,f). \text{Then } u^+ \in \mathcal{N}^{\Gamma}_{0,0,f} \text{ and}$$

$$m^{\tau}(0,0,f) \le \Big(\min_{x\in\Omega} \frac{\#\Gamma x}{f(x)^{(N-2)/2}}\Big) \frac{2}{N} S^{N/2}.$$

Hence

$$m^{\Gamma}(0,0,f) \le E_{0,0,f}(u^+) = \frac{1}{2} m^{\tau}(0,0,f) \le \Big(\min_{x\in\Omega} \frac{\#\Gamma x}{f(x)^{\frac{N-2}{2}}}\Big) \frac{1}{N} S^{N/2} = m^{\Gamma}(0,0,f).$$

Thus u^+ is a minimum of $E_{0,0,f}$ on $\mathcal{N}^{\Gamma}_{0,0,f}$, which contradicts (NE) . The corollary 3 . 8 implies

$$m^{\tau}(0,0,f) = \left(\min_{x\in\Omega} \frac{\#\Gamma x}{f(x)^{(N-2)/2}} \right) \frac{2}{N} S^{N/2}.$$

$\square$

 Then property (4 . 3) implies

$$u^{\pm} \in \mathcal{N}^{\Gamma}_{0,0,f} \cap E^{\eta}_{0,0,f} \quad \forall u \in \mathcal{N}^{\tau}_{0,0,f} \cap E^{2\eta}_{0,0,f},$$

so

$$\| u^{+} - \nu W_{\varepsilon,y} \| = \min_{W\in\Theta_0} \| u^{+} - W \| \Leftrightarrow \| u^{-} + \nu W_{\varepsilon,g\tau y} \| \quad = \min_{W\in\Theta_0} \| u^{-} - W \| . \quad (4.4)$$

Therefore ,

$$\beta^{\Gamma}(u^{+}) = \Gamma y \iff \beta^{\Gamma}(u^{-}) = \Gamma(g\tau y), \quad (4.5)$$

and

$$\beta^{\Gamma}(u^{+}) \neq \beta^{\Gamma}(u^{-}) \quad \forall u \in \mathcal{N}^{\tau}_{0,0,f} \cap E^{2\eta}_{0,0,f}. \quad (4.6)$$

Set

$$B_{\delta}(M)^{\tau} := \{z \in B_{\delta}(M) : Gz = \Gamma z\}.$$

Proposition 4 . 4 . *The map*

$$\beta^{\tau} : \mathcal{N}^{\tau}_{0,0,f} \cap E^{2\eta}_{0,0,f} \to (B_{\delta}(M) \setminus B_{\delta}(M)^{\tau})/\Gamma, \quad \beta^{\tau}(u) := \beta^{\Gamma}(u^{+}),$$

is well defined , continuous and $\mathbb{Z}/2-$ *equivariant ; i . e . ,*

$$\beta^{\tau}(-u) = \Gamma(g\tau y) \iff \beta^{\tau}(u) = \Gamma y.$$

Proo f − period If $u \in \mathcal{N}^{\tau}_{0},_{0,f} \cap E^{2\eta}_{0,0,f}$ and $\beta^{\tau}(u) = \Gamma y \in B_{\delta}(M)^{\tau}/\Gamma$ then $\beta^{\Gamma}(u^{+}) = \Gamma y =$
$\Gamma(g\tau y) = \beta^{\Gamma}(u^{-})$, this is a contradiction to (4 . 6) . We conclude that $\beta^{\tau}(u)$ *element − negationslash*
$B_{\delta}(M)^{\tau}/\Gamma$. The continuity and $\mathbb{Z}/2-$ equivariant properties follows by β^{Γ} ones . □

5 . Multiplicity of solutions

5 . 1 . **Lusternik - Schnirelmann theory .** An involution on a topological space X is a map $\varrho X : X \to X$, such that $\varrho X \circ \varrho X = id_X$. Given an involution we can define an action of $\mathbb{Z}/2$ on X and viceversa . The trivial action is given by the identity $\varrho X = id_X$, the action of $G/\Gamma \simeq \mathbb{Z}/2$ on the orbit space $\mathbb{R}^N/\Gamma$ where $G \subset O(N)$ and Γ is the kernel of an epimorphism $\tau : G \to \mathbb{Z}/2$, and the antipodal action $\varrho(u) = -u$ on $\mathcal{N}^{\tau}_{\lambda,\mu,f}$. A map $f : X \to Y$ is called $\mathbb{Z}/2-$ equivariant (or a $\mathbb{Z}/2-$ map) if $\varrho Y \circ f = f \circ \varrho X$, and two $\mathbb{Z}/2-$ maps , $f0, f1 : X \to Y$, are said to be $\mathbb{Z}/2-$ homotopic if there exists a homotopy $\Theta : X \times [0,1] \to Y$ such that $\Theta(x,0) = f0(x)$, $\Theta(x,1) = f1(x)$ and $\Theta(\varrho X^{x}, t) = \varrho Y^{\Theta}(x,t)$ for every $x \in X, t \in [0,1]$. A subset A of X is $\mathbb{Z}/2-$ equivariant if $\varrho X^{a} \in A$ for every $a \in A$.

Definition 5 . 1 . The $\mathbb{Z}/2-$ category of a $\mathbb{Z}/2-$ map $f : X \to Y$ is the smallest integer $k := \mathbb{Z}/2-$ cat (f) with following properties

(i) There exists a cover of $X = X_1 \cup \cdots \cup X_k$ by k open $\mathbb{Z}/2-$ invariant subsets , (i i) The restriction $f \mid X_i : X_i \to Y$ is $\mathbb{Z}/2-$ homotopic to the composition $\kappa_i \circ \alpha_i$ of a $\mathbb{Z}/2-$ map $\alpha_i : X_i \to \{yi, \varrho Y yi\}, yi \in Y$, and the inclusion $\kappa_i : \{yi, \varrho Y yi\} \to$

$$Y.$$

If not such covering exists , we define $\mathbb{Z}/2-$ cat $(f) := \infty$.

If A is a $\mathbb{Z}/2-$ invariant subset of X and $\iota : A \to X$ is the inclusion we write

$$\mathbb{Z}/2 - cat_X(A) := \mathbb{Z}/2 - \mathrm{cat}(\iota), \quad \mathbb{Z}/2 - cat_X(X) := \mathbb{Z}/2 - \mathrm{cat}(X).$$

Note that if $\varrho x = id_X$ then

$$\mathbb{Z}/2 - cat_X(A) := cat_X(A), \quad \mathbb{Z}/2 - \mathrm{cat}(X) := \mathrm{cat}(X),$$

are the usual Lusternik - Schnirelmann category (see [1 7 , definition 5 . 4]) .

Theorem 5.2. *Let $\phi : M \to \mathbb{R}$ be an even functional of class C^1, and M a submanifold of a Hilbert space of class C^2, symmetric with respect to the origin. If ϕ is bounded below and satisfies $(PS)_c$ for each $c \le d$, then ϕ has at least $\mathbb{Z}/2-$*cat (ϕ^d) *pairs critical points such that $\phi(u) \le d$.*

5.2. **Proof of Theorems.** We prove Theorem 2.3 only; the proof of Theorem 2.1 is analogous. Recall that if τ is the identity or an epimorphism then $\#(G/\Gamma)$ is 1 or 2.

Proof of Theorem 2.3. By Corollary 3.8, $E_{\lambda,\mu,f}$ satisfies $(PS)^\tau_\theta$ for

$$\theta < \min\{\#(G/\Gamma)\frac{\ell^\Gamma_f}{N}, \frac{\#(G/\Gamma)}{N} S_\mu^{N/2}\}.$$

By Lusternik-Schnirelmann theory $E_{\lambda,\mu,f}$ has at least $\mathbb{Z}/2-$ cat $(\mathcal{N}^\tau_{\lambda,\mu,f} \cap E^\theta_{\lambda,\mu,f})$ pairs $\pm u$ of critical points in $\mathcal{N}^\tau_{\lambda,\mu,f} \cap E^\theta_{\lambda,\mu,f}$. We are going to estimate this category for an appropriate value of θ.

Without lost of generality we can assume that $\delta \in (0,\delta_0)$, with δ_0 as in (4.2). Let $\eta > \frac{\ell^\Gamma_f}{N}$, $\mu^* \in (0, \underline{\quad}\mu)$ and $\lambda^* \in (0,\lambda_1)$ such that

$$(\frac{\bar\mu}{\bar\mu - \mu^*})^{N/2}(\frac{\lambda_1}{\lambda_1 - \lambda*})^{N/2} = \min\{2, \frac{N\eta}{\#(G/\Gamma)\ell^\Gamma_f}, \ell\frac{\ell^\Gamma_f}{\Gamma f - \delta\prime}\}.$$

By Lemma 3.3, if $u \in \mathcal{N}^\tau_{\lambda,\mu,f} \cap E^\theta_{\lambda,\mu,f}, \mu \in (0,\mu^*), \lambda \in (0,\lambda^*)$ we have

$$\begin{aligned} E_{0,0,f}(\pi_{0,0,f}(u)) &\le (\frac{\bar\mu}{\bar\mu-\mu})\frac{N}{2}(\frac{\lambda_1}{\lambda_1-\lambda})\frac{N}{2} E_{\lambda,\mu,f}(u) \\ &< (\frac{\bar\mu}{\bar\mu-\mu})\frac{N}{2}(\frac{\lambda_1}{\lambda_1-\lambda})\frac{N}{2}\#(G/\Gamma)\frac{\ell^\Gamma_f}{N} \\ &\le \#(G/\Gamma)\eta. \end{aligned}$$

Let β^τ be the $\tau-$ bariorbit function, defined in Proposition 4.4. Hence the com-position map

$$\beta^\tau \circ \pi_{0,0,f} : \mathcal{N}^\tau_{\lambda,\mu,f} \cap E^\theta_{\lambda,\mu,f} \to (B_\delta(M) \setminus B_\delta(M)^\tau)/\Gamma,$$

is a well defined $\mathbb{Z}/2-$ invariant continuous function.

By the [3, Proposition 3] using (F2) we can choose $\varepsilon > 0$ small enough and

$$\begin{aligned} \theta := \theta_\varepsilon &< \#(G/\Gamma)\frac{\ell^\Gamma_f}{N} \text{such that} \\ E_{\lambda,\mu,f}(\pi_{\lambda,\mu,f}(w^\tau_{\varepsilon,y})) \le \theta &< \#(G/\Gamma)\frac{\ell^\Gamma_f}{N}, \quad \forall y \in M^-_\delta, \\ \text{where} w^\tau_{\varepsilon,y} &= \varepsilon^\Gamma_{w,y} - \varepsilon^\Gamma_{w,g\tau y}, \tau(g\tau) = -1, \text{and} \\ \varepsilon^\Gamma_{w,y}(x) &= \sum_{[\gamma]\in\Gamma/\Gamma_y} f(y)^{(2-N)/4} U_{\varepsilon,\gamma y}(x)\varphi_{\gamma y}(x). \end{aligned}$$

Thus the map

$$\begin{aligned} \alpha\tau_\delta : M^-_{\tau,\delta}/\Gamma &\to \mathcal{N}^\tau_{\lambda,\mu,f} \cap E^\theta_{\lambda,\mu,f}, \\ \alpha^\tau\delta(\Gamma y) &:= \pi_{\lambda,\mu,f}(w^\tau_{\varepsilon,y}), \end{aligned}$$

 is a well defined $\mathbb{Z}/2-$ invariant continuous function . Moreover , $\beta^{\tau}(\pi_{0,0,f}(\alpha\tau_{\delta}(\Gamma y))) = \Gamma y$ for all $y \in M^{-}_{\tau,\delta}$. Therefore ,

$$\mathbb{Z}/2 - \mathrm{cat}(\mathcal{N}^{\tau}_{\lambda,\mu,f} \cap E^{\theta}_{\lambda,\mu,f}) \geq \mathrm{cat}_{((B_{\delta}(M)\backslash B_{\delta}(M)^{\tau})/\Gamma)}(M^{-}_{\tau,\delta}/\Gamma).$$

So (2 . 1 0) has at least

$$\mathrm{cat}_{((B_{\delta}(M)\backslash B_{\delta}(M)^{\tau})/G)}(M^{-}_{\tau,\delta}/G)$$

pairs $\pm u$ solution which satisfy

$$E_{\lambda,\mu,f}(u) < \#(G/\Gamma)\frac{\ell^{\Gamma}_{f}}{N}.$$

By the choice of λ^{*} and μ^{*} we have

$$(\frac{\bar{\mu}}{\bar{\mu}-\mu^{*}})^{N/2}(\frac{\lambda_1}{\lambda_1-\lambda*})^{N/2} \leq \frac{\ell^{\Gamma}_{f}}{\ell^{\Gamma}_{f}-\delta\prime}.$$

Then

$$\begin{aligned}\#(G/\Gamma)\frac{\ell^{\Gamma}_{f}-\delta'}{N} &\leq (\frac{\bar{\mu}-\mu}{\bar{\mu}})^{N/2}(\frac{\lambda_1-\lambda}{\lambda_1})^{N/2}\#(G/\Gamma)\frac{\ell^{\Gamma}_{f}}{N}\\ &\leq m^{\tau}(\lambda,\mu,f) \leq E_{\lambda,\mu,f}(u)\\ &= \frac{1}{N}\parallel u \parallel^{2}_{\lambda,\mu} < \#(G/\Gamma)\frac{\ell^{\Gamma}_{f}}{N}\end{aligned}$$

therefore

$$\#(G/\Gamma)\ell^{\Gamma}_{f} - \delta'' \leq \quad \parallel u \parallel 2\lambda, \mu < \#(G/\Gamma)\ell^{\Gamma}_{f}.$$

□

Proof of Theorem 2 . 4 . By Theorem 2 . 1 there exist λ and μ sufficiently close to zero such that the problem (2 . 1) has at least $\mathrm{cat}_{B_{\delta}(M)/\Gamma}(M^{-}_{\delta}/\Gamma)$ positive solutions such

$$\mathrm{that} E_{\lambda,\mu,f}(u) < \frac{\ell^{\Gamma}_{f}}{N}.$$

We will prove that $\frac{\ell^{\Gamma}_{f}}{N} < m^{\mathrm{Gamma-e}}(0,0,f)$. First suppose that $m^{e-Gamma}(0,0,f)$ does not achieve then by the hypothesis $m^{\mathrm{Gamma-e}}(0,0,f) = \frac{\ell^{e-Gamma}_{f}}{N} > \frac{\ell^{\Gamma}_{f}}{N}$. If $m^{\mathrm{Gamma-e}}(0,0,f)$ is achieved there exists $u \in \mathcal{N}^{\mathrm{Gamma-e}}_{0,0,f} \subset \mathcal{N}^{\Gamma}_{0,0,f}$ and

$$\frac{\ell^{\Gamma}_{f}}{N} = m^{\Gamma}(0,0,f) < m^{\mathrm{Gamma-e}}(0,0,f) = E_{0,0,f}(u).$$

By (3 . 4) there exist $\widehat{\lambda} \in (0,\lambda_1)$ and $\widehat{\mu} \in (0,\bar{\mu})$ such that for each $\lambda \in (0,\widehat{\lambda})$ and

$\mu \in (0,\widehat{\mu})$suchthat

$$\frac{\ell^{\Gamma}_{f}}{N} < m^{\tilde{\Gamma}}(0,0,f) \leq (\frac{\lambda_1}{\lambda_1-\lambda})^{N/2}(\frac{\mu}{-\mu} -\mu)^{N/2}m^{\tilde{\Gamma}}(\lambda,\mu,f).$$

Then

$$E_{\lambda,\mu,f}(u) < \frac{\ell_f^{\Gamma}}{N} < m^{\tilde{\Gamma}}(\lambda,\mu,f).$$

Therefore , u is not $\tilde{\Gamma}_-$ invariant solution . $\square$

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