International Journal of Pure and Applied Mathematics
Volume 102 No. 4 2015, 711-732
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v102i4.10

ijpam.eu

CONSUMPTION AND PORTFOLIO DECISIONS OF A RATIONAL AGENT THAT HAS ACCESS TO AN AMERICAN PUT OPTION ON AN UNDERLYING ASSET WITH STOCHASTIC VOLATILITY

Ma. Teresa V. Martínez-Palacios[1], Francisco Venegas-Martínez[2 §], José Francisco Martínez Sánchez[3]

[1]Escuela Superior de Apan
Universidad Autónoma del Estado de Hidalgo
Carretera Apan-Calpulalpan Km. 8
Col. Chimalpa, 43920 Apan, Hgo. MEXICO
[2]Escuela Superior de Economía
Instituto Politécnico Nacional (ESE-IPN)
Plan de Agua Prieta 66, Plutarco Elas Calles
Miguel Hidalgo, 11350 Ciudad de Mxico, D.F., MEXICO
[3]Escuela Superior de Apan
Universidad Autónoma del Estado de Hidalgo
Carretera Apan-Calpulalpan Km. 8
Col.Chimalpa, 43920 Apan, Hgo. MEXICO

Abstract: This paper is aimed at developing a model of a risk-averse rational consumer that has an initial wealth and faces the decision to allocate his wealth between consumption and investment in a portfolio of assets in a finite time horizon of stochastic length, so as to maximize his/her expected total utility. Particularly, the agent may invest in an American put option on an asset with stochastic volatility. Finally, the valuation of the American put option is carried out by using the Monte Carlo method.

AMS Subject Classification: 91B51, 62M10, 91G20, 65C05

Received: March 17, 2015

§Correspondence author

Key Words: continuous time stochastic optimal control, stochastic volatility, American contingent claims, Monte Carlo simulation

1. Introduction

The contingent reality faced by different economic agents participating in the various financial markets impacts their decision making on consumption and portfolio. In order to model this kind of decision making in risky environments, sophisticated mathematical tools have been developed in recent years with a boost up. Particular attention has been paid to the theoretical approach of dynamic stochastic general equilibrium models (DSGEM). Under this framework, there are diverse available models in the literature see, for example: [63], [19], [55], [85], [84], [50],and [56], among many others. Under the DSGEM framework, there are also other papers involving richer stochastic environments in which economic and financial variables are driven by mixed jump-diffusion processes; see, in this regard: [63], [61], [77], [78], [16] and [20].

One important issue in macroeconomic research consist in developing stochastic models that explain stylized facts on consumption and portfolio decisions, see, for instance: [31], [30], [45], [11], [78], [15], and [17], to name a few. Regarding pricing contingent claims of the American type a concept that provides consistency with decisions that can be made at any point in time in uncertain environments is that of stopping time process; see: [29], [69], [73], [85], [84], [61], [55], [80], [70], and [83], among others. For example, Shreve ([73]) analyzes the optimal exercise time of an American put option through the binomial model and defines the stopping time to be the first time at which the put option is equal to its intrinsic value. On the other hand, Bjrk ([85] and [84]) imposes a stopping time to avoid degenerated solutions provided nor a control constraint neither an inheritance function forbid the economic agent increase his utility at any level.

In most of the previous investigations, an essential assignment is to obtain prices of available financial assets and their derivatives. This literature is, actually, vast and varied; see, in this respect: [13], [65], [31], [30], [68], [81], [36], [35], [34], [84], [20], [18], [15], [72], [23], [26], [27], and [70], among others. A peculiarity in the research of Cruz-Ak and Venegas-Martnez ([72] and [21]) is that the authors obtain prices of derivatives on several assets and commodities.

A valuation approach frequently used, even when market risk cannot be hedged, is provided by DSGEM. In this regard see, for instance, the pioneering work from Lucas ([66]) and Cox, Ingersoll and Ross ([31]). More recent papers on this issue are [11], [37], [16], [20], [21], [72], and [23]. A common feature of these investigations is a finite and deterministic time horizon; this subject will be generalized in the present research.

Generalizations of the Black and Scholes' (1973) partial differential equation are the most widely used mathematical models to value derivatives. This allows, depending on the nature of the imposed boundary conditions, obtaining the theoretical price of different derivatives available in financial markets (exchanges and over-the-counter). In this context, it is important to extend the B&S model by introducing stochastic volatility of the underlying asset. For example, Grajales and Perez ([4]) estimate the parameters for the stock indices with the family of ARCH (discrete time) models and with the empirical model of Wilmott and Oztukel ([58]). They assume that the distribution of volatility of the returns is lognormal. Meanwhile, Sierra ([27]) extends the B&S equation towards stochastic volatility as in Hull and White ([36]) but considering fractional Brownian motion where the price of the underlying and its volatility are uncorrelated. Also, Venegas-Martnez ([19]) models the kurtosis and skew of financial series by using the combination of Brownian motion with Poisson jumps. Other investigations using stochastic volatility are: [36], [10], [75], [79], [8], [3], [48], [1], and [40], to name a few. For example Herzel ([75]) extends the B&S model assuming that the volatility of the share price can jump from one value to another in a random instant of time, and finds analytical solutions for pricing options. Leon and Serna ([1]) use the semiparametric models of Corrado-Su ([5]) and Jondeau-Rockinger ([9]), as well as the model of mixture of lognormal distributions of Bahra ([2]) comparing them with the seminal B&S model. Their results indicate that Corrado-Su and Jondeau-Rockinger improve the original B&S model. Meanwhile Britten-Jones and Neuberger ([40]) characterize the processes of continuous prices that are consistent with actual prices of options extending the work of Derman and Kani ([8]), Dupire ([3]) and Rubinstein ([48]) since they only consider processes with deterministic volatility. Among the works that allow the underlying price to jump, it may be mentioned the works of Merton ([64]), and Venegas-Martínez ([16], [22] and [19]), among others.

The research for pricing derivatives is very extensive; see, for example: [65], [28], [12], [42], [43], [6], [32], [34], [38], [82], [74], [54], [7], [76], [67], [24], [57],

[44], [34], and [70], among others. In the case of American claims, part of the valuation problem is to identify the exercise border that maximizes the value of the option for the owner of the contract; see, in this regard, [65], [28], and [49]. Although it is best to have a closed formula for American options, this is not often available, and the research effort has been focused on developing approximate methods; see, [82], [76], [46], and [32]. Meanwhile, Whaley ([67]), and Barone-Adesi and Whaley ([24]) developed an approximate analytic formula with a very fast performance compared to other methods. On the other hand, Kou and Wang ([74]) show that using a jump-diffusion model they obtain an analytical approximation for a finite horizon. Some other researchers adopt an interpolation scheme to price an American option; for instance, [41], [34], and [82]. Within the methods that follow approximated schemes based on exact representations with free boundary are those of an integral representation as in Broadie and Detemple ([41]) and Ju ([54]). Finally, a common feature among several different methods is to use the stopping time process as a fundamental tool for the valuation of American derivatives, see: [60], [70], [34], [82], [29], [84], and [83], among others.

The distinguishing features of this research, with respect to the above investigations are: 1) the planning horizon is finite but with stochastic length, 2) the stopping time process not only avoids degenerate solutions but also is helpful in modeling an American derivative, and 3) the valuation of the American option is carried out through the Monte-Carlo method.

This paper is organized as follows. The following section describes the theoretical framework needed to develop the model. Section 3 deals with the agent's intertemporal budget constraint. Section 4 introduces the stopping time process that prevents the problem of producing degenerate solutions ([84]). Here, the continuous time optimal control problem with a finite stochastic planning horizon is stated. Section 5 provides the solution to the proposed optimal control problem. Section 6 presents the first order conditions for an interior solution. Section 7 employs, for consistency, the verification theorem of the stochastic dynamic programming. Section 8 provides the price of an American put option on an AMX stock by using the Monte Carlo method. Finally, Section 9 presents the conclusions highlighting the limitations and indicating those extensions that will be considered in future research.

2. Assets and Returns

Suppose that the agent is allowed to invest his wealth in assets with and without risk. Hence, a portion of his wealth may be invested as savings in a bank that pays a risk-free interest rate $r > 0$ that is continuously compounded. Thus, the balance of the investment at time t is $B_t = B_0 e^{rt}$, which can be expressed, equivalently, by the basic firs-order differential equation

$$dR_{B_t} = \frac{dB_t}{B_t} = r dt. \tag{1}$$

2.1. The Underlying Asset with Stochastic Volatility

Other part of his wealth may be invested into two risky assets, a share whose volatility is stochastic and an American put option on that share. It is assumed that the stock price follows the following stochastic differential equation:

$$dR_{S_t} = \frac{dS_t}{S_t} = \mu dt + \sigma_t dW_t, \tag{2}$$

where W_t is a Wiener processes or Brownian motion, defined on a fixed probability space with its augmented filtration $(\Omega, \mathcal{F}, (\mathcal{F}_t^W)_{t\in[0,T]}, \mathbb{P})$, $\mu \in \mathbb{R}$ is the trend parameter, and $\sigma_t^2 = V_t$ is the variance (or squared volatility). The volatility of the underlying asset is driven by a second geometric Brownian motion with the following stochastic differential equation:

$$dR_{V_t} = \frac{dV_t}{V_t} = \alpha dt + \beta dZ_t, \tag{3}$$

where α is the trend of the variance and $\beta > 0$ stands for the variance volatility. Both quantities are known. The process dZ_t is a Brownian motion defined on a fixed probability space with augmented filtration $(\Omega, \mathcal{F}, (\mathcal{F}_t^Z)_{t\in[0,T]}, \mathbb{P})$. Particularly, it is assumed that the aforementioned Brownian motions are uncorrelated, that is,

$$\mathrm{Cov}\,(dW_t, dZ_t) = 0. \tag{4}$$

2.2. American Put Option

The agent has access to a third asset, an American put option on stock with price $A_t = A_t(S_t, V_t, t)$, and dynamics driven by:

$$dR_{A_t} = \frac{dA_t}{A_t} = \mu_{A_t} dt + \sigma_{1A_t} dW_t + \sigma_{2A_t} dZ_t, \tag{5}$$

where,

$$\mu_{A_t} = \frac{1}{A_t}\left[\frac{\partial A_t}{\partial t} + \frac{\partial A_t}{\partial S_t}\mu S_t + \frac{\partial A_t}{\partial V_t}\alpha V_t + \tfrac{1}{2}\left(\frac{\partial^2 A_t}{\partial S_t^2}\sigma_t^2 S_t^2 + \frac{\partial^2 A_t}{\partial V_t^2}\beta^2 V_t^2\right)\right] dt$$

$$\sigma_{1A_t} = \frac{1}{A_t}\frac{\partial A_t}{\partial S_t}\sigma S_t \qquad \text{and} \qquad \sigma_{2A_t} = \frac{1}{A_t}\frac{\partial A_t}{\partial V_t}\beta V_t. \tag{6}$$

The proportions of wealth allocated to the risky asset and the American put option in the portfolio at time t will be denoted by ω_{1t} y ω_{2t}, respectively. Thus, the proportion of wealth in the portfolio allocated to the riskless asset is $1-\omega_{1t}-\omega_{2t}$. Variable c_t denotes the consumption rate, and is restricted to self-financing strategies of consumption-investment. It is further suppose that the agent lives in a world where continuous negotiations are possible, without fall into any costs by commissions to brokers or tax payments to government. Finally, short sales are allowed and unlimited.

3. Agents Intertemporal Budget Constraint

Let X_t be the consumer wealth at time t, thus the stochastic dynamics of marginal wealth is given by:

$$dX_t = X_t\omega_{1t}dR_{S_t} + X_t\omega_{2t}dR_{A_t} + X_t\left(1-\omega_{1t}-\omega_{2t}\right)dR_{B_t} - c_t dt,$$

that is,

$$\begin{aligned} dX_t = X_t&\left(r + \omega_{1t}\left(\mu - r\right) + \omega_{2t}\left(\mu_{A_t} - r\right) - \frac{c_t}{X_t}\right) dt + \\ &+ X_t\left(\omega_{1t}\sigma_t + \omega_{2t}\sigma_{1A_t}\right) dW_t + X_t\left(\omega_{2t}\sigma_{2A_t}\right) dZ_t, \end{aligned} \tag{7}$$

equivalently,

$$\frac{dX_t}{X_t} = \mu_X dt + \sigma_{1X} dW_t + \sigma_{2X} dZ_t \tag{8}$$

where

$$\begin{aligned} \mu_X &= \left(r + \omega_{1t}\left(\mu - r\right) + \omega_{2t}\left(\mu_{A_t} - r\right) - \frac{c_t}{X_t}\right) \\ \sigma_{1X} &= \omega_{1t}\sigma_t + \omega_{2t}\sigma_{1A_t} \qquad \text{and} \qquad \sigma_{2X} = \omega_{2t}\sigma_{2A_t}. \end{aligned} \tag{9}$$

4. Stopping Time and Stochastic Optimal Control

Consider an economic agent with a finite planning horizon given by the time interval $[0, T]$ with a random final value for T . Assume that at time $t = 0$ the agent has an initial wealth X_0 and faces the decision to assign his/her wealth between consumption and investment, so as to his/her wealth stays no negative over planning horizon. The consumer maximizes his total expected discounted utility from consumption. Suppose that the satisfaction index of the agent is given by:

$$\mathbb{E}\left[\int_0^T F\left(t, c_t\right) dt + \Phi\left(X_T\right) |\mathcal{F}_0\right]$$

Where F is the discounted utility function for consumption and Φ stands for the retirement function at time T. The quantity Φ measures the usefulness of having a reserve at the end of the period, and $\mathcal{F}_0$ is the relevant available information at time $t = 0$.

Suppose now that the agent can borrow unlimited resources and invest in assets so his wealth could become zero at some point and even negative. Thus, T becomes a stopping time. To deal with this difficulty, we define the restricted domain $D = [0, T] \times \{x|\, x > 0\}$. Let us now define the function

$$\tau = \min\left[\inf\left\{t > 0|\, X_t = 0\right\}, T\right].$$

The interpretation is that when the process of wealth attains the domain boundary, i.e., if wealth is zero, then the activity is completed and there is no retirement function, so it is natural for Φ to become zero.

4.1. Setting the Stochastic Optimal Control Problem

The utility maximization problem of the rational consumer as a stochastic optimal control problem can be stated as:

$$\begin{aligned}
&\underset{\omega_{1t}, \omega_{2t}, c_t}{\text{Maximize}}\, \mathbb{E}\left[\int_0^\tau F(t, c_t) dt\, |\mathcal{F}_0\right], \\
&dX_t = X_t \mu_X dt + X_t \sigma_{1X} dW_t + X_t \sigma_{2X} dZ_t \\
&X_0 = x_0, \\
&c_t \geq 0, \forall\, t \geq 0.
\end{aligned} \tag{10}$$

5. Dynamic Programming, Hamilton Jacobi Bellman (HJB) Equation

To solve the problem stated in (10) and find the optimal proportions of wealth allocated in the investment portfolio as well as optimal consumption, we define the value function associated with the optimization problem as follows:

$$\begin{aligned} J\left(X_t, V_t, t\right) &= \max_{\omega_{1t},\omega_{2t}\in\mathbb{R},0\leq c_s|_{[t,\tau]}} \mathbb{E}\left[\int_t^{\tau} F\left(c_s, s\right) ds \,|\mathcal{F}_t\right] \\ &= \max_{\omega_{1t},\omega_{2t}\in\mathbb{R},0\leq c_s|_{[t,\tau]}} \mathbb{E}\left[\int_t^{t+dt} F\left(c_s, s\right) ds + \int_{t+dt}^{\tau} F\left(c_s, s\right) ds \,|\mathcal{F}_t\right]. \end{aligned} \tag{11}$$

After applying the mean value theorem of integral calculus to the first term and using a recursive relationship in the second term, we obtain

$$\begin{aligned} J\left(X_t, V_t, t\right) = \max_{\omega_{1t},\omega_{2t}\in\mathbb{R},0\leq c_s|_{[t,t+dt]}} \mathbb{E}\Big\{ & F(c_t, t)dt + o(dt) \\ & + J(X_t + dX_t, V_t + dV_t, t + dt)\Big|\mathcal{F}_t\Big\}. \end{aligned}$$

After using the Taylor series expansion in the second term, we get

$$\begin{aligned} J\left(X_t, V_t, t\right) = \max_{\omega_{1t},\omega_{2t}\in\mathbb{R},0\leq c_s|_{[t,t+dt]}} \mathbb{E}\Big\{ & F(c_t, t)dt + o(dt) + J(X_t, V_t, t) \\ & + dJ(X_t, V_t, t) + o(dt)\Big|\mathcal{F}_t\Big\}. \end{aligned}$$

Consequently,

$$0 = \max_{\omega_{1t},\omega_{2t}\in\mathbb{R},0\leq c_s|_{[t,t+dt]}} \mathbb{E}\Big\{F\left(c_t, t\right) dt + o\left(dt\right) + dJ\left(X_t, V_t, t\right)\Big|\mathcal{F}_t\Big\}.$$

If we use Its lemma to $dJ(X_t, V_t, t)$, it follows

$$\begin{aligned} 0 = \max_{\omega_{1t},\omega_{2t}\in\mathbb{R},0\leq c_s|_{[t,t+dt]}} \mathbb{E}\Bigg\{ & F\left(c_t, t\right) dt + o\left(dt\right) + \Bigg[\frac{\partial J\left(X_t, V_t, t\right)}{\partial t} + \\ & + \frac{\partial J\left(X_t, V_t, t\right)}{\partial X_t} X_t \mu_X + \frac{\partial J\left(X_t, V_t, t\right)}{\partial V_t} V_t \alpha + \\ & + \frac{1}{2}\frac{\partial^2 J\left(X_t, V_t, t\right)}{\partial X_t^2} X_t^2 \left(\sigma_{1X}^2 + \sigma_{2X}^2\right) + \frac{1}{2}\frac{\partial^2 J\left(X_t, V_t, t\right)}{\partial V_t^2} V_t^2 \beta^2 + \end{aligned}$$

$$+\frac{\partial^2 J\left(X_t,V_t,t\right)}{\partial X_t \partial V_t} X_t V_t \beta\sigma_{2X}\Bigg] dt + \frac{\partial J\left(X_t,V_t,t\right)}{\partial X_t} X_t (\sigma_{1X} dW_t + \sigma_{2X} dZ_t) +$$
$$+\frac{\partial J\left(X_t,V_t,t\right)}{\partial V_t} V_t \beta dZ_t \Big| \mathcal{F}_t \Bigg\}.$$

Subsequently, taking into account that $dW_t \sim N\left(0,dt\right)$ and $dZ_t \sim N\left(0,dt\right)$, the expected value of the above equation is

$$0 = \max_{\omega_{1t},\omega_{2t}\in\mathbb{R},0\leq c_s|_{[t,t+dt]}} \Bigg\{ F\left(c_t,t\right)dt + o\left(dt\right) + \Bigg[\frac{\partial J\left(X_t,V_t,t\right)}{\partial t} +$$
$$+\frac{\partial J\left(X_t,V_t,t\right)}{\partial X_t} X_t \mu_X + \frac{\partial J\left(X_t,V_t,t\right)}{\partial V_t} V_t \alpha +$$
$$+\frac{1}{2}\frac{\partial^2 J\left(X_t,V_t,t\right)}{\partial X_t^2} X_t^2\left(\sigma_{1X}^2+\sigma_{2X}^2\right) + \frac{1}{2}\frac{\partial^2 J\left(X_t,V_t,t\right)}{\partial V_t^2} V_t^2\beta^2 +$$
$$+\frac{\partial^2 J\left(X_t,V_t,t\right)}{\partial X_t \partial V_t} X_t V_t \beta\sigma_{2X}\Bigg] \Big| \mathcal{F}_t \Bigg\}.$$

The above expression is now divided by dt and after taking the limit when $dt \to 0$, it is obtained

$$0 = \max_{\omega_{1t},\omega_{2t}\in\mathbb{R},0\leq c_s|_{[t,t+dt]}} \Bigg\{ F\left(c_t,t\right) + \frac{\partial J\left(X_t,V_t,t\right)}{\partial t} + \frac{\partial J\left(X_t,V_t,t\right)}{\partial X_t} X_t \mu_X +$$
$$+\frac{\partial J\left(X_t,V_t,t\right)}{\partial V_t} V_t \alpha + \frac{1}{2}\frac{\partial^2 J\left(X_t,V_t,t\right)}{\partial X_t^2} X_t^2\left(\sigma_{1X}^2+\sigma_{2X}^2\right) +$$
$$+\frac{1}{2}\frac{\partial^2 J\left(X_t,V_t,t\right)}{\partial V_t^2} V_t^2\beta^2 + \frac{\partial^2 J\left(X_t,V_t,t\right)}{\partial X_t \partial V_t} X_t V_t \beta\sigma_{2X} \Bigg\}.$$

In the above equation we impose the corresponding boundary conditions

$$
\begin{aligned}
0 = & \max_{\omega_{1t},\omega_{2t}\in\mathbb{R},0\leq c_s|_{[t,t+dt]}} \left\{ F\left(c_t,t\right) + \frac{\partial J\left(X_t,V_t,t\right)}{\partial t} + \right. \\
& + \frac{\partial J\left(X_t,V_t,t\right)}{\partial X_t} X_t \mu_X + \frac{\partial J\left(X_t,V_t,t\right)}{\partial V_t} V_t \alpha + \\
& + \frac{1}{2}\frac{\partial^2 J\left(X_t,V_t,t\right)}{\partial X_t^2} X_t^2\left(\sigma_{1X}^2+\sigma_{1X}^2\right) + \frac{1}{2}\frac{\partial^2 J\left(X_t,V_t,t\right)}{\partial V_t^2} V_t^2\beta^2 + \\
& \left. + \frac{\partial^2 J\left(X_t,V_t,t\right)}{\partial X_t \partial V_t} X_t V_t \beta\sigma_{2X} \right\}, \\
& J\left(X,V,T\right)=0, \\
& J\left(0,V,t\right)=0.
\end{aligned}
\tag{12}
$$

It is worth pointing out that the above boundary conditions have incorporated the stopping time.

5.1. Utility Function

Suppose that utility has the functional form $F\left(c_t,t\right)=e^{-\rho t}U\left(c_t\right)$, where $U(c_t)$ is a member of the family utility functions HARA ([60] and [53]). In what follows, we chose the following consumption function

$$
F\left(c_t,t\right)=\frac{e^{-\delta t}}{\delta}\ln\left(c_t\right).
$$

5.2. First-Order Conditions

By assuming an interior maximum and making the appropriate substitutions, we obtain the HJB condition:

$$
\begin{aligned}
0 = & \frac{e^{-\delta t}}{\delta}\ln\left(c_t\right) + \frac{\partial J\left(X_t,V_t,t\right)}{\partial t} + \\
& + \frac{\partial J\left(X_t,V_t,t\right)}{\partial V_t} V_t\alpha + \frac{1}{2}\frac{\partial^2 J\left(X_t,V_t,t\right)}{\partial V_t^2} V_t^2\beta^2 + \\
& + \frac{\partial J\left(X_t,V_t,t\right)}{\partial X_t} X_t\left(r+\omega_{1t}\left(\mu-r\right)+\omega_{2t}\left(\mu_{A_t}-r\right)-\frac{c_t}{X_t}\right) + \\
& + \frac{1}{2}\frac{\partial^2 J\left(X_t,t\right)}{\partial X_t^2} X_t^2\left[\omega_{1t}^2\sigma_t^2+\omega_{2t}^2\sigma_{1A_t}^2+2\omega_{1t}\sigma_t\omega_{2t}\sigma_{1A_t}+\omega_{2t}^2\sigma_{2A_t}^2\right] +
\end{aligned}
\tag{13}
$$

$$+\frac{\partial^2 J\left(X_t,V_t,t\right)}{\partial X_t \partial V_t}\beta V_t X_t \omega_{2t}\sigma_{2A_t}.$$

Hence, the first order conditions are:

$$c_t=\left[\frac{\partial J\left(X_t,V_t,t\right)}{\partial X_t}\delta e^{\delta t}\right]^{-1},$$

$$\omega_{1t}=-\frac{\left[\frac{\partial J(X_t,V_t,t)}{\partial X_t}+\frac{\partial^2 J(X_t,t)}{\partial X_t^2}X_t^2\left(\sigma_{1A_t}\omega_{2t}\sigma_t\right)X_t\left(\mu-r\right)\right]}{\frac{\partial^2 J(X_t,t)}{\partial X_t^2}X_t^2\sigma_t^2} \tag{14}$$

$$\omega_{2t}=-\frac{\left[\frac{\partial J\left(X_t,V_t,t\right)}{\partial X_t}X_t\left(\mu_{A_t}-r\right)+\frac{\partial^2 J\left(X_t,V_t,t\right)}{\partial X_t \partial V_t}X_tV_t\beta\sigma_{2A_t}+ +\frac{\partial^2 J\left(X_t,t\right)}{\partial X_t^2}X_t^2\left(\sigma_{1A_t}\omega_{1t}\sigma_t\right)\right]}{\frac{\partial^2 J(X_t,t)}{\partial X_t^2}\left(\sigma_{1A_t}^2+\sigma_{2A_t}^2\right)}$$

6. Solution of the HJB Equation

To choose a candidate solution for $J(X_t,V_t,t)$ that satisfies the HJB condition, we set

$$J\left(X_t,V_t,t\right)=\frac{e^{-\delta t}}{\delta}\left[g\left(V_t,t\right)+\ln\left(x_t\right)\right], \tag{15}$$

along with $g\left(V_t,T\right)=0$. Then, we have Maxwell's equations:

$$\frac{\partial J(X_t,V_t,t)}{\partial t}=-e^{-\delta t}\left[\ln\left(X_t\right)+g(V_t,t)\right] +\frac{e^{-\delta t}}{\delta}\left(\frac{\partial g\left(V_t,t\right)}{\partial t}\right), \tag{16a}$$

$$\frac{\partial J(X_t,V_t,t)}{\partial X_t}=\frac{e^{-\delta t}}{\delta}X_t^{-1},\quad \frac{\partial J(X_t,V_t,t)}{\partial V_t}=\frac{e^{-\delta t}}{\delta}\frac{\partial g(V_t,t)}{\partial V_t} \tag{16b}$$

$$\frac{\partial^2 J(X_t,V_t,t)}{\partial X_t^2}=-\frac{e^{-\delta t}}{\delta}X_t^{-2},\quad \frac{\partial^2 J(X_t,V_t,t)}{\partial V_t^2}=\frac{e^{-\delta t}}{\delta}\frac{\partial^2 g(V_t,t)}{\partial V_t^2} \tag{16c}$$

$$\frac{\partial^2 J(X_t,V_t,t)}{\partial V_t \partial X_t} = 0 = \frac{\partial^2 J(X_t,V_t,t)}{\partial X_t \partial V_t}. \tag{16d}$$

6.1. Optimal Decisions

After substituting the obtained values in (16) into (14), we obtain:

$$\hat{c}_t = X_t, \tag{17}$$

$$\begin{aligned} \hat{\omega}_{1t} &= \frac{(\mu - r)\left(\sigma_{1A_t}^2 + \sigma_{2A_t}^2\right) + (r - \mu_{A_t})\,\sigma_t \sigma_{1A_t}}{\sigma_{2A_t}^2 \sigma_t^2} \\ \hat{\omega}_{2t} &= \frac{(\mu_{A_t} - r)\,\sigma_t + (-\mu + r)\,\sigma_{1A_t}}{\sigma_t \sigma_{2A_t}^2}. \end{aligned} \tag{18}$$

Notice that $\hat{c}$ is linear in wealth, and consumption becomes a random variable, a situation that is according with reality. The quantities $\hat{\omega}_{1t}$ y $\hat{\omega}_{2t}$ are the optimal portfolio proportions and they remain constant.

7. Pricing American Options by Using Monte Carlo Method

In this section by using the Monte Carlo method, and based on the assumptions of section 2, we deal with the valuation of an American put option on a stock that not pays dividends and whose price process is supposed to be driven by the geometric Brownian motion:

$$d\,(\ln S_t) = \left(r - \tfrac{1}{2}\sigma_t^2\right) dt + \sigma_t dW_t, \tag{19}$$

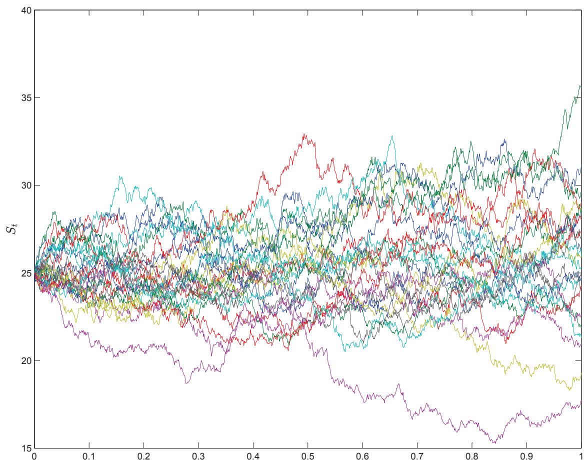
or in discrete version

$$S_{t+\Delta t} = S_t e^{\left\{\left(r - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}\varepsilon\right\}}, \tag{20}$$

where ε is a standard normal random variable. Equation (20) allows the simulation of prices taking as initial value the closing price of a share and generating a random number, which will lead to the price at the next time, so it continues until the end of the time horizon obtaining the full simulated series. The following figure show 25 simulated trajectories of prices on AMX stock; so they can be seen.

In order to compute the value of the American put option, we use the following algorithm:

Figure 1: Monte Carlo Simulation of 25 trajectories.



Source: Authors own elaboration.

a) Simulate the process of daily share prices, starting with the closing price of a certain day;

b) Realize n simulations;

c) Determine the maximum of each simulated series and the intrinsic value of the American put option;

d) Compute the present value of the intrinsic value.

e) Calculate the average of the previous values, which determines the value of the American put option price.

With the aim to generate the normal random numbers, the Box-Muller Method is used, which states that to generate values $\varepsilon \sim \mathcal{N}(0,1)$, it may be used

$$\varepsilon = \sqrt{-2\ln(U_1)\cos(2\pi U_2)}, \text{ or } \varepsilon = \sqrt{-2\ln(U_1)\sin(2\pi U_2)} \tag{21}$$

with $U_1, U_2 \sim \mathrm{U}[0,1]$.

Table 1 shows the initial parameter values and the obtained prices of American put and call options on AMX stock, based on 2000 realizations (a few realizations are computed for straightforwardness, at least 10,000 are recommended). The expiration time is one month, and the variance is 20%.

Table 1: Initial parameter values and prices of American put and call options.

Monte Carlo simulation	
Initial price of the asset	24.79
Expiration date	T=30/360
Volatility	20%
Step time	1/360
Interes rate	3.40%
Exercise Price	25
American call option price	0.73556
American put option price	0.031428

Souce: Author's own elaboration.

Notice that in the proposed stochastic optimal control approach, the planning horizon is stochastic, which requires for the derivative to be exercised at any time within the planning horizon (this characterizes the American option). In the valuation problem, we need not only to determine the value of the option at each instant, but also to know whether the option is exercise for each value of S_t. Usually, this is done by setting a critical value S^* for each of these moments ([51]). Particularly, for the valuation through Monte Carlo method, we have considered the border of exercise S^* defined by the stopping time, namely, when the maximum share price is attained in the planning horizon.

8. Conclusions

Under the assumption that the planning horizon is finite but with stochastic length and using a stopping time process for an American claim, we provide an alternative pricing procedure by using the Monte Carlo method. The proposed scheme was applied for pricing American put and call options on AMX stock.

One of the possible generalizations of this research is to use new functional forms for the stochastic dynamics of the price of the underlying asset including unexpected jumps in its volatility and incorporating a stochastic interest rate, which will be considered in the future. It is also important to think in the future about that the price of the underlying asset can be driven by Markov-modulated diffusion process.

References

[1] A. León, G. Serna, Modelos alternativos de valoración de opciones sobre acciones: Una aplicación al mercado espaol, *Cuadernos Económicos del ICE*, **69**, (2005), 33-50.

[2] B. Bahra, Implied risk-neutral probability density functions from option prices: theory and application, Working Paper no. 66, *Bank of England*, London (1997).

[3] B. Dupire, Pricing with a smile, *Risk*, **7**, No. 2, (1994), 32-39.

[4] C. A. Grajales, F. O. Prez, Modelos discretos y continuos para estimar la densidad de probabilidad de la volatilidad estocástica de los rendimientos de series financieras, *Cuadernos de Administración*, **21**, No. 36, (2008), 113-132.

[5] C. J. Corrado, T. Su, Skewness and Kurtosis in S &P 500 Index Returns Implied by Option Prices, *The Journal of Financial Research*, **XIX**, No. 2, (1996a), 175-192.

[6] C. Rogers, Monte Carlo valuation of American options, *Mathematical Finance*, **12**, No. 3, (2002), 271-286.

[7] D. Jin-Chuan, S. Jean-Guy, American Option Pricing under GARCH by a Markov Chain Approximation, *Journal of Economic Dynamics and Control*, **25**, No. 11, (2001), 1689-1718.

[8] E. Derman, I. Kani, The Volatility Smile and Its Implied Tree, *Quantitative Strategies Research Notes*, (1994).

[9] E. Jondeau, M. Rockinger, Gram-Charlier Densities, *Journal of Economic Dynamics and Control*, **25**, (2001), 1457-1483.

[10] E. M. Stein, J. C. Stein, Stock Price Distributions with Stochastic Volatilty: An Analytic Approach, *Review of Financial Studies*, No. 4, (1991), 727-752.

[11] E. R. Grinols, S. J. Turnovsky, Risk, the Financial Market and Macroeconomic Equilibrium, *Journal of Economic Dynamics and Control*, **17**, (1993), 1-36.

[12] F. A. Longstaff, E. S. Schwartz, Valuing American Options by Simulation: A Simple Least-Squares Approach, *The Review of Financial Studies*, **14**, No. 1, (2001), 113-147.

[13] F. Black, M. Scholes, The Pricing of Options and Corporate Liabilities, *The Journal of Political Economy*, **81**, No. 3, (1973), 637-654.

[14] F. Venegas-Martnez, A. Rodrguez-Nava, Consumo y decisiones de portafolio en ambientes estocsticos: un marco terico unificador, *Ensayos Revista de Economa*, **XXVII** , No. 2, (2009), 29-64, Mxico

[15] F. Venegas-Martnez, F. Ortiz-Arango, Impacto de la poltica fiscal en un ambiente con inflacin estocstica: un modelo de control ptimo, *Morfismos*, **14**, No.1, (2010), 51-68.

[16] F. Venegas-Martnez, Opciones Cobertura y Procesos de Difusin con Saltos: Una Aplicacin a los Ttulos de GCARSO, *Estudios Econmicos*, **16**, No. 32, (2001a), 203- 226.

[17] F. Venegas-Martnez, Poltica fiscal, estabilizacin de precios y mercados incompletos, *Estudios econmicos*, **20**, (2005), 3-25.

[18] F. Venegas-Martnez, Real Options on Consumption in a Small Open Monetary conomy: A Stochastic Optimal Control Approach, *Morfismos*, **11** , No. 1, (2007), 37-52.

[19] F. Venegas-Martnez, *Riesgos financieros y econmicos, productos derivados y decisiones econmicas bajo incertidumbre*, Segunda edicin, Cengage, Mxico (2008).

[20] F. Venegas-Martnez, Stochastic Temporary Stabilization: Undiversifiable Devaluation and Income Risks, *Economic Modelling*, **23**, No. 1, (2006), 157-173.

[21] F. Venegas-Martnez, Un modelo estocstico de equilibrio general para valuar derivados y bonos, *Econoquantum*, **6**, No. 1, (2009), 111-120.

[22] F. Venegas-Martnez, Una gua completa para economistas en la valuacin de Opciones, *Gaceta Economa*, Ao 6, No. 12, (2001b), 155-212.

[23] G. ngeles, F. Venegas-Martnez, Valuacin de opciones sobre ndices burstiles y determinacin de la estructura de plazos de la tasa de inters de un modelo de equilibrio general, *Investigacin Econmica*, **69**. No. 271, (2010), 43-80.

[24] G. Barone-Adesi, R. E. Whaley Efficient Analytic Approximation of American Option Values, *The Journal of Finance*, **XLII**, No. 2, (1987), 301-320.

[25] G. Sierra, Aplicacin del mtodo H-J-B para la modelacin estocstica de la volatilidad en series con persistencia: El caso del IPC en Mxico, *Eseconoma*, Nm. 20, (2008), 73-99.

[26] G. Sierra, Procesos Hurst y Movimiento Browniano Fraccional en Mercados Fractales: Valuacin y aplicacin a los derivados y Finanzas, *Concurso Mexder*, (2007).

[27] G. Sierra, Relacin de procesos de persistencia estimados a travs de los coeficientes Hurst y volatilidad implcita en el mercado mexicano, Stable URL *http://www.ecorfan.org/series.php*, (2012).

[28] H. M. Taylor, Evaluating a call option and optimal timing strategy in the stock market, *Management Science*, **14**, No. 1, (1967), 111120.

[29] I. Karatzas, S. Shreve, *Brownian Motion and Stochastic Calculus*, Springer, New York, USA (1998).

[30] J. C. Cox, J. E. Ingersoll, S. A. Ross, A Theory of the Term Structure of Interest Rates, *Econometrica*, **53**, No. 2, (1985b), 363-384.

[31] J. C. Cox, J. E. Ingersoll, S. A. Ross, An Intertemporal General Equilibrium Model of Asset Prices, *Econometrica*, **53**, No. 2, (1985a), 385-408.

[32] J. C. Cox, S. A. Ross, M. Rubinstein, Option Pricing: A Simplified Approach, *Journal of Financial Economics*, **7**, No. 3, (1979), 229-263.

[33] J. Detemple, S. Feng, W. Tian, The Valuation of American Call Options on the Minimum of Two Dividend-Paying Assets, *The Annals of Applied Probability*, **13**, No. 3, (2003), 953-983.

[34] J. Detemple, T. Weidong, The Valuation of American Options for a Class of Diffusion Processes, *Management Science*, **48**, No. 7, (2002), 917-937.

[35] J. Hull, A. White, One-Factor Interest-Rate Models and the Valuation of Interest-Rate Derivative Securities, *The Journal of Financial and Quantitative Analysis*, **28**, No. 2, (1993), 235-254.

[36] J. Hull, A. White, The Pricing of Options on Assets with Stochastic Volatilities, *The Journal of Finance*, **42**, No. 2, (1987), 281-300.

[37] K. Schmedders, Computing equilibria in the general equilibrium model with incomplete asset markets, *Journal of Economic Dynamics and Control*, **22** (8-9), (1998), 1375-1401.

[38] L. Stentoft, Pricing American Options when the Underlying Asset follows GARCH processes, *Journal of Empirical Finance*, **12**, No. 4, (2005), 576-611.

[39] M. B. Haugh, L. Kogan, Pricing American Options: A Duality Approach, *Operations Research*, **52**, No. 2, (2004), 258270.

[40] M. Britten-Jones, A. Neuberger, Option Prices, Implied Price Processes, and Stochastic Volatility, *The Journal of Finance*, **55**, No. 2, (2000), 839-866.

[41] M. Broadie, J. Detemple, American Option Valuation: New Bounds, Approximations, and a Comparison of Existing Method, *Review of Financial Studies*, **9**, No. 4, (2004a), 1211-1250.

[42] M. Broadie, J. Detemple, Option Pricing: Valuation Models and Applications *Management Science*, **50**, No. 9, (2004b), 1145-1177.

[43] M. Broadie, P. Glasserman, H. Zachary, Probabilistic Constrained Optimization: Methodology and Applications: Chapter: Pricing American Options by Simulation Using a Stochastic Mesh with Optimized Weights, *Kluwer Academic Publishers*, (2000), 32-50.

[44] M. C. Fu, S. B. Laprise, D. B. Madan, Y. Su, R. Wu, Pricing American Options: A Comparison of Monte Carlo Simulation Approaches, *Journal of Computational Finance*, **4**, No. 3, (2001), 39-88.

[45] M. Gertler, E. L. Grinols, Monetary Randomness and Investment, *Journal of Monetary Economics*, **10**, (1982), 239-258.

[46] M. J. Brennan, E. S. Schwartz, The Valuation of American Put Options, *The Journal of Finance*, **32**, No. 2, (1977), 449-462.

[47] M. Kijima, *Stochastic Processes with Applications to Finance*, Chapman and Hall/CRC, Boca Raton Florida, USA (2003).

[48] M. Rubinstein, Implied Binomial Trees, *Journal of Finance*, **49**, No. 3, (1994), 771-818.

[49] M. T. V. Martnez-Palacios, A. Snchez-Daza, F. Venegas-Martnez, Valuacin de opciones americanas: un enfoque de control ptimo estocstico en un horizonte finito con fecha final aleatoria, *Anlisis Econmico*, **XXVII**, No. 64, (2012), 165-183.

[50] M. T. V. Martnez-Palacios, F. Venegas-Martnez, Control ptimo estocstico en la enseanza de la economa matemtica, *Educacin Matemtica*, **23**, No. 3, (2011), 147-181.

[51] M. T. V. Martnez-Palacios, F. Venegas-Martnez, Un modelo macroeconmico con agentes de vida finita y estocstica: cobertura de riesgo de mercado con derivados americanos, *Economa, Teora y Prctica*, No. 41, (2014), 71-106.

[52] M. Wickens, *Macroeconomic Theory: A Dynamic General Equilibrium Approach*, Princeton and Oxford University Press (2008).

[53] N. Hakansson, Optimal Investment and Consumption Strategies Under Risk for a Class of Utility Functions, *Economtrica*, **38**, No. 5, (1970), 587-607.

[54] N. Ju, Pricing American Option by Approximating Its Early Exercise Boundary as a Multipiece Exponential Function, *The Review of Financial Studies*, **11**, No. 3, (1998), 627-646.

[55] O. Hernndez-Lerma, F. Venegas-Martnez, Toma de decisiones de agentes racionales con procesos markovianos: avances recientes en economa y finanzas, *El Trimestre Econmico*, **LXXIX(4)**, No. 316, (2012), 733-779.

[56] P. Huyn, *Continuous-time Stochastic Control and Optimization with Financial Applications*, Springer-Verlag, Berlin Heidelberg (2009).

[57] P. Huyn, Optimal Stopping, Free Boundary, and American Option in a Jump-Diffusion Model, *Mathematics and Statistics: Applied Mathematics & Optimization*, **35**, No. 2, (1997), 145-164.

[58] P. Wilmott, A. Oztukel, Uncertain parameters an empirical stochastic volatility model and confidence limits, International Journal of Theoretical and Applied Finance, Vol. 1, No. 1, (1998), 175-189.

[59] P. Wilmott, S. Howison J. Dewynne, *The Mathematics of Financial Derivatives, A Student Introduction*, Cambridge University Press, New York, USA (1999).

[60] R. C. Merton, *Continuous-Time Finance*, Basil Blackwell, Cambridge, Massachusetts (1990).

[61] R. C. Merton, Continuous-Time Finance, *Review of economics and statistics*, **51**, No. 2, (1992), 247-257.

[62] R. C. Merton, Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case, *Review of economics and statistics*, **51**, No. 2, (1969), 247-257.

[63] R. C. Merton, Optimum Consumption and Portfolio Rules in a Continuous-Time Model, *Journal of Economic Theory*, **3**, No. 4, (1971), 373-413.

[64] R. C. Merton, Option Pricing when Underlying Stock Returns are Discontinuous, *Journal of Financial Economics*, **3**, No. 1-2, (1976), 125-144.

[65] R. C. Merton, Theory of Rational Option Pricing, *Bell Journal of Economics*, **4**, No. 1, (1973), 141-183.

[66] R. E. Jr. Lucas, Asset Prices in an Exchange Economy, *Econometrica*, **46**, No. 6, (1978), 1429-1445. Stable URL: http://links.jstor.org/sici?sici=0012 9682

[67] R. E. Whaley, On the Valuation of American Call Options on Stocks with Known Dividends, *Journal of Financial Economics*, **9**, No. 2, (1981), 207-211.

[68] R. Geske, K. Shastri, Valuation by Approximation: A Comparison of Alternative Option Valuation Techniques, *The Journal of Financial and Quantitative Analysis*, **20**, No. 1, (1985), 45-71.

[69] R. Kohn, *Partial Differential Equations for Finance*, course notes (2003).

[70] R. Kohn, *Partial Differential Equations for Finance*, section 6, course notes (2011).

[71] R. Whaley, Valuation of American Futures Options: Theory and Empirical Tests, The Journal of Finance, Vol. 41, No. 1, (1986), 127-150.

[72] S. Cruz-Ake, F. Venegas-Martnez, Productos derivados sobre bienes de consumo, *Econo Quantum*, **6**, No. 1, (2010), 25-54.

[73] S. E. Shreve, *Stochastic Calculus for Finance I*, Springer, New York, USA (2005).

[74] S. G. Kou, H. Wang, Option Pricing Under a Double Exponential Jump Diffusion Model, *Management Science*, **50**, No. 9, (2004), 11781192.

[75] S. Herzel, A Simple Model for Option Pricing with Jumping Stochastic Volatility, *International Journal of Theoretical and Applied Finance*, **1**, No. 4, (1998), 487-505.

[76] S. Ikonen, J. Toivanen, Efficient Numerical Methods for Pricing American Options under Stochastic Volatility, *Numerical Methods for Partial Differential Equations*, **24**, No. 1, (2008), 104126.

[77] S. J. Turnovsky, *Methods of macroeconomic dynamics*, 2a. ed., MIT Press.,USA (2000).

[78] S. J. Turnovsky, W. T. Smith, Equilibrium consumption and precautionary savings in a stochastically growing economy, *Journal of Economic Dynamics and Control*, **30**, (2006), 243278.

[79] S. L. Heston, A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, *The review of financial studies*, **6**, No. 2, (1993), 327-343.

[80] S. P. Sethi, G. L. Thompson, *Optimal Control Theory: Aplications to Management Sciences and EconomicS*, Second Edition, Springer, New York, USA (2000).

[81] S. Thomas, Y. Ho, S-B. Lee, Term Structure Movements and Pricing Interest Rate Contingent Claims, *The Journal of Finance*, **41**, No. 5, (1986), 1011-1029.

[82] S. Villeneuve A. Zanette, Parabolic ADI Methods for Pricing American Options on Two Stocks, Mathematics and Statistics: Mathematics of Operations Research, Vol. 27, No. 1, (2002), 121-149.

[83] S. Villeneuve, On Threshold Strategies and the Smooth-Fit Principle for Optimal Stopping Problems, *Journal of Applied Probability*, **44**, No. 1, (2007), 181-198.

[84] T. Bjrk, *Arbitrage Theory in Continuous Time*, Second Edition, Oxford University Press, New York, USA (2004).

[85] T. Bjrk, J. Myhrman, M. Persson, Optimal consumption with stochastic prices in continuous time, *Journal of Applied Probability*, **24**, No. 1, (1987), 35-47.

[86] T. S. Ho, R. Stapleton, M. G. Subrahmanyam, The Valuation of American Options with Stochastic Interest Rates: A Generalization of the Geske-Johnson Technique, *The Journal of Finance*, **52**, No. 2, (1997), 827-840.

[87] V. Fernndez, Teora de opciones: una sntesis, *Revista de Anlisis Econmico*, **14**, No.2, (1999), 87-116.