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On the 3-restricted edge connectivity of permutation graphs

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ABSTRACT

An edge cut W of a connected graph G is a k-restricted edge cut if $G - W$ is disconnected, and every component of $G - W$ has at least k vertices. The k-restricted edge connectivity is defined as the minimum cardinality over all k-restricted edge cuts. A permutation graph is obtained by taking two disjoint copies of a graph and adding a perfect matching between the two copies. The k-restricted edge connectivity of a permutation graph is upper bounded by the so-called minimum k-edge degree. In this paper some sufficient conditions guaranteeing optimal k-restricted edge connectivity and super k-restricted edge connectivity for permutation graphs are presented for $k = 2, 3$.

1. Introduction

Throughout this work only undirected simple graphs without loops or multiple edges are considered. Unless stated otherwise, we follow [10] for terminology and definitions.

Let $G = (V(G), E(G))$ be a graph with set of vertices $V := V(G)$ and set of edges $E := E(G)$. A subset W of edges is an *edge cut* if $G - W$ is not connected. It is widely known that $\lambda(G) \leq \delta(G)$, where $\lambda(G)$ is the standard edge connectivity and $\delta(G)$ is the minimum degree of G. A graph G is *maximally edge connected* if $\lambda(G) = \delta(G)$.

The restricted edge connectivity was proposed by Esfahanian and Hakimi [11] who denoted it by $\lambda'(G)$. For a connected graph G the *restricted edge connectivity* is defined as the minimum cardinality of a set W of edges such that $G - W$ is not connected and W does not contain the set of incident edges to any vertex of the graph, then $G - W$ does not contain isolated vertices. The restricted edge connectivity has been studied under the name of *super edge connectivity*. This is a stronger measure of connectivity than the standard edge connectivity, and was proposed by Boesch [7] and Boesch and Tindell [8]. A graph is *super edge connected* or *super-λ*, if every minimum edge cut consists of a set of edges incident with one vertex. See [7,8,14] for more details. Clearly $\lambda'(G) > \delta(G)$ is a sufficient and necessary condition for G to be super edge connected.

Inspired by the definition of conditional connectivity introduced by Hararay [16], Fàbrega and Fiol [12,13] proposed the concept of k-restricted edge connectivity (where k is a nonnegative integer) as follows. An edge cut W is called a *k-restricted edge cut* if every component of $G - W$ has at least $k + 1$ vertices. In this paper we adopt the following definition. An edge cut W is called a *k-restricted edge cut* if every component of $G - W$ has at least k vertices, where $k \geq 1$. Assuming that G has k-restricted edge cuts, the *k-restricted edge connectivity* of G, denoted by $\lambda_{(k)}(G)$, is defined as the minimum cardinality over all k-restricted edge cuts of G. From the definition, we immediately have that if $\lambda_{(k)}(G)$ exists, then $\lambda_{(i)}(G)$ exists for any $i < k$ and $\lambda_{(i)}(G) \leq \lambda_{(k)}(G)$. Observe that any edge cut of G is a 1-restricted edge cut and $\lambda_{(1)}(G)$ is just the standard connectivity $\lambda(G)$. Furthermore, the restricted edge connectivity $\lambda'(G)$ defined in [11] is $\lambda'(G) = \lambda_{(2)}(G)$.

For a graph G and a permutation π of V, the *permutation graph* G^{π} is defined by taking two disjoint copies of G and adding a matching between these two copies such that each vertex v of one copy of G is joined with vertex $\pi(v)$ of the other

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copy. Examples of permutation graphs include some generalized Petersen graphs, hypercubes, and prisms. Observe that the cartesian product graph $K_2 \times G$ can be viewed as the permutation graph G^{id}, where *id* is the identity permutation. It must also be pointed out that a permutation graph can be understood within the frame of *product graphs* $H * G$, since G^{π} can be written as $K_2 * G$ (see [6] for the definition of this product of graphs). Due to their structure, permutation graphs provide a model for large-scale parallel processing systems. Moreover, permutation graphs can be seen as suitable models for building larger interconnection networks from smaller ones without increasing significantly their maximum transmission delay, in such a way that these larger networks are highly fault-tolerant. In this regard, several results for the connectivity of permutation graphs are given in [2,15,17,20,21]; see also [3,4] for the connectivity of product graphs $H * G$.

In this work we study the k-restricted edge connectivity of permutations graphs. We present bounds when $k \in \{2, 3\}$, generalizing some results contained in [2]. The article is organized as follows. In Section 2 we recall some definitions and present some basic results about the k-restricted edge connectivity. Section 3 is devoted to presenting the aforementioned bounds for the k-restricted edge connectivities of permutation graphs.

2. Notation and preliminary results

Let $G = (V, E)$ be a graph. Given a proper subset X of V, let $\omega(X) = [X, V \setminus X]$ denote the set of edges with one end in X and the other end in $V \setminus X$. Let $G[X]$ denote the subgraph induced by X. For every nonnegative integer k, the *minimum k-edge degree* is defined as follows

$$\xi_{(k)}(G) = \min\{|\omega(X)| : |X| = k, G[X] \text{ is connected}\}.$$

Clearly, $\xi_{(1)}(G) = \delta(G)$ and $\xi_{(2)}(G) = \xi(G) = \min\{d(u) + d(v) - 2 : uv \in E(G)\}$, usually known as the *minimum edge degree of G*.

A graph G is said to be $\lambda_{(k)}$*-connected* if k-restricted edge cuts exist. In [11] was shown that $\lambda_{(2)}(G)$ exists and $\lambda_{(2)}(G) \le \xi(G)$ if G is not a star and its order is at least 4. For $k = 3$, it was shown [9,19] that except for a special class of graphs named *flowers*, 3-restricted edge cuts exist and $\lambda_{(3)}(G) \le \xi_{(3)}(G)$ for any connected graph G with order at least 7. Following Ou [19], a graph F of order $n \ge 2k$ is called a *flower* if it contains a cut-vertex s such that every component of $F - s$ has order at most $k - 1$. Furthermore, Zhang and Yuan [24] showed that if G is a connected graph of minimum degree δ and order $n \ge 2(\delta + 1)$ that is not isomorphic to any $G^*_{m,\delta}$ (where $G^*_{m,\delta}$ consists of m disjoint copies of K_δ and a new vertex u adjacent to all the vertices in those copies) and $k \le \delta + 1$, then G has k-restricted edge cuts and $\lambda_{(k)}(G) \le \xi_{(k)}(G)$.

In this paper we restrict ourselves to $\lambda_{(k)}$-connected graphs G with $\lambda_{(k)}(G) \le \xi_{(k)}(G)$. A graph G is said to be *optimally k-restricted edge connected* if it is $\lambda_{(k)}$-connected and $\lambda_{(k)}(G) = \xi_{(k)}(G)$. In the rest of the paper, an optimally k-restricted edge connected graph is said to be for short $\lambda_{(k)}$*-optimal*. Several results assuring optimal k-restricted edge connectivity for graphs with small diameter were obtained in [1,5].

A k-restricted edge cut $\omega(X) = [X, V \setminus X]$ is called $\lambda_{(k)}$*-cut* if $|\omega(X)| = |[X, V \setminus X]| = \lambda_{(k)}(G)$. It is clear for any $\lambda_{(k)}$-cut $\omega(X)$ that $G - \omega(X)$ has just two connected components. If $\omega(X)$ is a $\lambda_{(k)}$-cut of G, then $X \subset V$ is called a *k-fragment* of G. It is clear that if X is a k-fragment of G, then so is $V \setminus X$ and the subgraphs induced by X and by $V \setminus X$ are both connected. Let $a_k(G) = \min\{|X| : X \text{ is a } k\text{-fragment of } G\}$. Obviously, $k \le a_k(G) \le |V|/2$. A k-fragment X is called a *k-atom* of G if $|X| = a_k(G)$. Xu and Xu [23] proved that every $\lambda_{(2)}$-optimal graph other than a triangle has $a_2(G) = 2$. Bonsma et al. [9] proved that a $\lambda_{(3)}$-connected graph is $\lambda_{(3)}$-optimal if and only if $a_3(G) = 3$. Inspired by these results we present a result for guaranteeing $a_k(G) = k$ assuming certain additional conditions.

Theorem 2.1. *Let G be a $\lambda_{(k)}$-connected graph with $\lambda_{(k)}(G) \le \xi_{(k)}(G)$. Then G is $\lambda_{(k)}$-optimal if $a_k(G) = k$. Moreover, $a_k(G) = k$ follows when G is $\lambda_{(k)}$-optimal and some of the following assertions hold for its minimum degree δ and its girth g:*

(i) $\delta \ge 2k - 1$.
(ii) $\delta \ge k + 1$ *and* $g \ge k + 1$.

Proof. If $X \subset V(G)$ is a k-atom with cardinality $|X| = a_k(G) = k$, then $\lambda_{(k)}(G) = |\omega(X)| \ge \xi_{(k)}(G)$, yielding $\lambda_{(k)}(G) = \xi_{(k)}(G)$. Next, suppose $\lambda_{(k)}(G) = \xi_{(k)}(G)$, and let $X \subset V(G)$ be such that $|X| = k$, $G[X]$ is connected, and $|\omega(X)| = \xi_{(k)}(G)$. Let C be any component of $G - \omega(X)$ distinct from $G[X]$, and consider a vertex $z \in V(C)$. As z can be adjacent to at most k vertices of X, if $\delta \ge 2k - 1$, there must exist at least $d(z) - k \ge \delta - k \ge k - 1$ neighbors of z in C, hence $|V(C)| \ge k$. Then $\omega(X)$ is a $\lambda_{(k)}$-cut, yielding $a_k(G) \le |X| = k$, hence $a_k(G) = k$. Furthermore, if $g \ge k + 1$ holds for the girth and $\delta \ge k + 1$, then z can be adjacent to at most 2 vertices of X. Then if $\delta \ge k + 1$ there must exist at least $d(z) - 2 \ge \delta - 2 \ge k - 1$ neighbors of z in C, hence $|V(C)| \ge k$. As before we have $a_k(G) = k$. □

The concept of super restricted edge connected graph G, or super-$\lambda_{(2)}$ was proposed by Li and Li [18] and by Wang [22]. A graph G is *super restricted edge connected* if G is $\lambda_{(2)}$-optimal and the deletion of every minimum 2-restricted edge cut of G isolates an edge. Clearly, if G is super restricted edge connected, then $a_2(G) = 2$.

The concept of super restricted edge connected can be generalized for any $\lambda_{(k)}$-connected graph G as follows.

Definition 2.2. A graph G on n vertices is *super k-restricted edge connected*, or *super-*$\lambda_{(k)}$, if G is $\lambda_{(k)}$-optimal and the deletion of every $\lambda_{(k)}$-cut isolates a component with k vertices; that is, if every k-fragment X has cardinality $|X| \in \{k, n - k\}$.

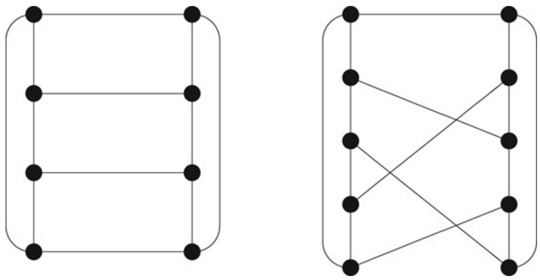

Fig. 1. Non super-$\lambda_{(k)}$-graph G for which $\lambda_{(k)}(G) = \xi_{(k)}(G)$, for $k = 2$ (left) and $k = 3$ (right).

Observe that a super-$\lambda_{(k)}$ graph G has $a_k(G) = k$. Moreover, G has $\lambda_{(k)}(G) = \xi_{(k)}(G)$, but the converse is not true as the two examples depicted in Fig. 1 show.

Theorem 2.3. *Let G be a $\lambda_{(k)}$-connected graph such that $\lambda_{(k)}(G) \le \xi_{(k)}(G)$ and $\lambda_{(k+1)}(G)$ exists. Then G is super-$\lambda_{(k)}$ if and only if $\lambda_{(k+1)}(G) > \xi_{(k)}(G)$.*

Proof. Let G be super-$\lambda_{(k)}$, that is, $\lambda_{(k)}(G) = \xi_{(k)}(G)$ and every k-fragment of G has cardinality k or $n - k$, where n is the order of G. Suppose $\lambda_{(k+1)}(G) \le \xi_{(k)}(G)$, and let W be a $\lambda_{(k+1)}$-cut of G. Then $|W| = \lambda_{(k+1)}(G)$ and $G - W$ consists of exactly two connected components (due to minimality of W) with vertex sets X and $X^* = V \setminus X$, with $|X|, |X^*| \ge k + 1$ and $W = \omega(X) = \omega(X^*)$. Notice that W is also $\lambda_{(k)}$-cut, because $\lambda_{(k+1)}(G) \le \xi_{(k)}(G) = \lambda_{(k)}(G)$ yields $|W| = \lambda_{(k+1)}(G) = \lambda_{(k)}(G)$, since clearly $\lambda_{(k)}(G) \le \lambda_{(k+1)}(G)$. Therefore, X and X^* are k-fragments of G with $|X|, |X^*| \ge k + 1$, which contradicts that G is super-$\lambda_{(k)}$.

For the converse suppose that G is not super-$\lambda_{(k)}$ and $\lambda_{(k+1)}(G) > \xi_{(k)}(G)$. Then there exists a $\lambda_{(k)}$-cut $\omega(X)$ such that neither X nor $V \setminus X$ has cardinality k (hence $|X|, |V \setminus X| \ge k + 1$). Therefore, $\omega(X)$ is also a $(k + 1)$-restricted edge cut, and $\xi_{(k)}(G) < \lambda_{(k+1)}(G) \le |\omega(X)| = \lambda_{(k)}(G)$, contradicting $\lambda_{(k)}(G) \le \xi_{(k)}(G)$. □

The following result states a relationship between two different minimum k-edge degrees.

Lemma 1. *Let G be a connected graph with minimum degree δ and minimum k-edge degree $\xi_{(k)}(G)$ with $k \le \delta + 1$. Then for every $k \ge 2$ and for every $j \in \{0, \dots, k\}$ it follows that*

$$\xi_{(k)}(G) \ge \xi_{(k-j)}(G) + j\delta - 2jk + j(j + 1).$$

Proof. Let $X \subset V(G)$ be such that $|X| = k$, $G[X]$ is connected, and $\xi_{(k)}(G) = |\omega(X)|$. Notice that there exists some $x \in X$ such that $G[X - x]$ is still connected. Then

$$\begin{aligned}\xi_{(k)}(G) &= |\omega(X)| \\ &= |\omega(X - x)| - d_{G[X]}(x) + d_{G-(X-x)}(x) \\ &= |\omega(X - x)| - 2d_{G[X]}(x) + d_G(x) \\ &\ge \xi_{(k-1)}(G) - 2(k - 1) + \delta.\end{aligned}$$

By means of an iterative application of this inequality, for $j \in \{0, \dots, k\}$ we have $\xi_{(k)}(G) \ge \xi_{(k-j)}(G) + j(\delta - 2k + j + 1)$. □

As a consequence of Theorem 2.3 and Lemma 1 we obtain the following result.

Theorem 2.4. *Let $k \ge 1$ and let G be a $\lambda_{(k+1)}$-optimal graph with minimum degree $\delta \ge 2k + 1$. Then G is super-$\lambda_{(k-t)}$ for every $t = 0, \dots, k - 1$.*

Proof. According to the hypothesis on G we have $\lambda_{(k+1)}(G) = \xi_{(k+1)}(G)$. Therefore Lemma 1 together with the hypothesis $\delta \ge 2k + 1$ allows us to deduce that

$$\lambda_{(k+1)}(G) = \xi_{(k+1)}(G) \ge \xi_{(k)}(G) + \delta - 2(k + 1) + 2 > \xi_{(k)}(G).$$

Thus, Theorem 2.3 implies that G is super-$\lambda_{(k)}$, hence the result is true for $t = 0$. Since G is super-$\lambda_{(k)}$, then $\lambda_{(k)}(G) = \xi_{(k)}(G)$. Again Lemma 1 together with the hypothesis $\delta \ge 2k + 1$ allows us to deduce that

$$\lambda_{(k)}(G) = \xi_{(k)}(G) \ge \xi_{(k-1)}(G) + \delta - 2k + 2 > \xi_{(k-1)}(G).$$

As before Theorem 2.3 implies that G is super-$\lambda_{(k-1)}$. Repeating this reasoning we obtain the desired result. □

3. *k*-Restricted edge connectivity of permutation graphs

From now on, we denote the two copies of G in the permutation graph G^π by G_1 and G_2, and call *cross edges* the edges joining vertices of G_1 and G_2. Notice that $\delta(G^\pi) = \delta(G) + 1$, $\Delta(G^\pi) = \Delta(G) + 1$ hold for the minimum and maximum degrees, respectively. Next, we obtain a first result concerning the minimum k-edge degree of permutation graphs.

Lemma 2. *Let $k \geq 2$ and let G be a graph of minimum k-edge degree $\xi_{(k)}(G)$. For any permutation π of $V(G)$ it follows that*

$$\xi_{(k)}(G^\pi) \leq \xi_{(k)}(G) + k.$$

Moreover, if the minimum degree of G is $\delta(G)$, then

$$\xi(G^\pi) \geq 2\delta(G) \quad \text{and} \quad \xi_{(3)}(G^\pi) \geq \min\{\delta(G) + \xi(G) + 1, \xi_{(3)}(G) + 3\}.$$

Proof. Let G_1 and G_2 denote the two copies of G^π. Let $X \subset V(G_1)$ be such that $|X| = k$, $G[X]$ is connected and $|w_{G_1}(X)| = \xi_{(k)}(G)$. Then $\xi_{(k)}(G^\pi) \leq |w_{G^\pi}(X)| = \xi_{(k)}(G) + k$.

Let $Y \subset V(G^\pi)$ be such that $|Y| = k$, $G^\pi[Y]$ is connected and $|w_{G^\pi}(Y)| = \xi_{(k)}(G^\pi)$. Let us write $Y = Y_1 \cup Y_2$ with $Y_1 \subset V(G_1)$, $Y_2 \subset V(G_2)$. If $Y_1 = \emptyset$, it is clear that $\xi_{(k)}(G^\pi) = |w_{G^\pi}(Y_2)| = |w_{G_2}(Y_2)| + k \geq \xi_{(k)}(G) + k$, hence $\xi_{(k)}(G^\pi) \geq \xi_{(k)}(G) + k$. Then suppose that $0 < r = |Y_1| \leq |Y_2| = k - r$. In this case

$$\begin{aligned}\xi_{(k)}(G^\pi) &= |w_{G_1}(Y_1)| + |w_{G_2}(Y_2)| + |[Y_1, V(G_2) \setminus Y_2]| + |V(G_1) \setminus [Y_1, Y_2]| \\ &\geq \xi_{(r)}(G) + \xi_{(k-r)}(G) + k - 2r,\end{aligned}$$

because there are at most r cross edges joining vertices of Y_1 and Y_2. If $k = 2$, then $r = 1$ and $\xi_{(2)}(G^\pi) = \xi(G^\pi) \geq 2\xi_{(1)}(G) = 2\delta(G)$. If $k = 3$, then $r = 1$ and $\xi_{(3)}(G^\pi) \geq \xi_{(1)}(G) + \xi_{(2)}(G) + 1 = \delta(G) + \xi(G) + 1$. Therefore, $\xi_{(3)}(G^\pi) \geq \min\{\delta(G) + \xi(G) + 1, \xi_{(3)}(G) + 3\}$. □

The following theorem generalizes a result contained in [2] concerning the lower bound for the restricted edge connectivity of any permutation graph G^π.

Theorem 3.1. *Let G be a connected graph on $n \geq 6$ vertices and minimum degree $\delta(G) \geq 3$. Then for $k = 2, 3$ and for any permutation π, G^π is $\lambda_{(k)}$-connected and*

$$\min\{n, 2\lambda_{(k)}(G), \lambda_{(k)}(G) + \delta(G), \xi_{(k)}(G^\pi)\} \leq \lambda_{(k)}(G^\pi) \leq \xi_{(k)}(G^\pi).$$

Proof. Notice that G and G^π are $\lambda_{(k)}$-connected for $k = 2, 3$ because neither of them is a flower since $\delta(G^\pi) > \delta(G) \geq 3$. Therefore, for $k = 2, 3$, $\lambda_{(k)}(G) \leq \xi_{(k)}(G)$ and $\lambda_{(k)}(G^\pi) \leq \xi_{(k)}(G^\pi)$ [9,11].

Let us recall that G_1, G_2 stand for the two disjoint copies of G used to construct G^π and let $W \subset E(G^\pi)$ be a $\lambda_{(k)}$-cut, that is, $|W| = \lambda_{(k)}(G^\pi)$. Hence $G^\pi - W$ consists of exactly two connected components, H, H^* such that $|V(H)| \geq k$ and $|V(H^*)| \geq k$. Observe that $w(V(H)) = w(V(H^*)) = W = [V(H), V(H^*)]$. Notice that if $|V(H)| = k$, then $|W| = \lambda_{(k)}(G^\pi) \geq \xi_{(k)}(G^\pi)$ and the result holds. Let us denote by M the set of edges of G^π which connect vertices of G_1 with vertices of G_2. If $W = M$ the result is again true since $\lambda_{(k)}(G^\pi) = |M| = n$. Let us show next that the result also holds in case $k = 3$, $V(H) = \{u, v, u', w'\}$, with $uv, uu', u'w' \in E(G^\pi)$, $u, v \in V(G_1)$, $u', w' \in V(G_2)$. Indeed, if $B = \{u, v, u'\}$, we can write

$$\begin{aligned}|w_{G^\pi}(V(H))| &= |w_{G^\pi}(B)| + d_{G^\pi}(w') - 2|[\{w'\}, B]| \\ &\geq |w_{G^\pi}(B)| + d_{G^\pi}(w') - 4 \\ &\geq |w_{G^\pi}(B)| \\ &\geq \xi_{(3)}(G^\pi),\end{aligned}$$

after taking into account that $|[\{w'\}, B]| \leq 2$.

Thus we assume for the rest of the proof that $|V(H)| \geq k + 1$, $|V(H^*)| \geq k + 1$, $W \neq M$, and when $k = 3$, neither H nor H^* is a cycle of length four or a path of length three of exactly two vertices in G_1 and exactly two vertices in G_2. For the remaining cases we write heretofore $W = W_1 \cup W_M \cup W_2$, with $W_1 \subset E(G_1)$, $W_M \subset M$, $W_2 \subset E(G_2)$.

Notice that if $W_i \neq \emptyset$ then W_i is an edge cut of G_i due to the minimality of W. We claim next that every component of $G_i - W_i$ has cardinality at least k. On the contrary, assume that some component of $G_1 - W_1$ or of $G_2 - W_2$ has at most $k - 1$ vertices. Let C be such a component, chosen so that no other component of $(G_1 - W_1) \cup (G_2 - W_2)$ has fewer vertices than C, and (in case two or more components have this minimum order) with the minimum possible number of components of $(G_1 - W_1) \cup (G_2 - W_2)$ to which C is linked in $G^\pi - W$. Without loss of generality, assume that C is a component of $G_1 - W_1 \subset H$ satisfying these conditions. As G^π is $\lambda_{(k)}$-connected it follows that there exists a vertex $u \in V(C)$ such that the cross edge uu' is not in W_M. Let us see now that all components of $H - V(C)$ have at least k vertices. Suppose first that $|V(C)| = 1$, $V(C) = \{u\}$. Let $F = H - u$, which is connected as vertex u is only adjacent in H to vertex u'. In this case, $|V(F)| = |V(H)| - 1 \geq k$. Notice that $|V(C)| = 1$ holds when $k = 2$, hence we can suppose $k = 3$, $V(C) = \{u, v\}$, and C is linked in $G^\pi - W$ to at most two components C^*, C^{**} (not necessarily distinct) of $(G_1 - W_1) \cup (G_2 - W_2)$ (in fact, of $G_2 - W_2$), $|V(C^*)| \geq 2$, $|V(C^{**})| \geq 2$. We are clearly done if $|V(C^*)| \geq 3 = k$ and $|V(C^{**})| \geq 3 = k$. If $|V(C^*)| = 2$ and

$C^* \neq C^{**}$, by the way C has been chosen it follows that C^* is linked in $G^\pi - W$ to some other component of $G_1 - W_1$ different from C, hence C^* is contained in some component of $H - V(C)$ of cardinality at least $3 = k$ (and we can proceed similarly when $|V(C^{**})| = 2$ and $C^* \neq C^{**}$). Furthermore, if $C^* = C^{**}$ and $|V(C^*)| = 2$, H is either a cycle of length four or a path of length three, which contradicts our assumption. Once we have seen that every component of $H - V(C)$ has order at least k, it follows that the set of edges

$$W^* = (W \cup \{ww' : w \in V(C),\, w' \in V(G_2),\, ww' \in E(H) \setminus W_M\}) \setminus \omega_{G_1}(V(C))$$

is a k-restricted edge cut of G^π. But W^* has cardinality $|W^*| \leq |W| + |V(C)| - |\omega_{G_1}(V(C))| \leq |W| - |V(C)| \leq |W| - 1$ (because $|\omega_{G_1}(V(C))| \geq 2|V(C)|$ since $\delta(G_1) \geq 3 \geq k$ and $|V(C)| \geq 1$), an absurdity. We conclude that if $W_i \neq \emptyset$ then W_i is indeed a k-restricted edge cut of G_i, hence $|W_i| \geq \lambda_{(k)}(G)$.

Therefore, when both $W_1, W_2 \neq \emptyset$, then $\lambda_{(k)}(G^\pi) = |W| \geq |W_1| + |W_2| \geq 2\lambda_{(k)}(G)$, and the result holds. Hence we may assume $W_2 = \emptyset$ and in this case $V(H) \subset V(G_1)$ and $3 \leq k+1 \leq |V(H)| = |W_M|$. Since W_1 is a k-restricted edge cut of G_1 and $W_2 = \emptyset$, we have

$$\lambda_{(k)}(G^\pi) = |W| = |W_1| + |W_M| \geq \lambda_{(k)}(G) + |V(H)|. \tag{1}$$

First observe that if $|V(H)| \geq \delta(G)$ then from (1) we have $\lambda_{(k)}(G^\pi) \geq \lambda_{(k)}(G) + \delta(G)$, and the result holds. Therefore we assume $k + 1 \leq |V(H)| \leq \delta(G) - 1$. Let $|V(H)| = r \geq k + 1$, then by Lemma 1 we have

$$|W_1| \geq \xi_{(r)}(G) \geq \xi_{(k)}(G) + (r - k)(\delta(G) - r - k + 1).$$

If $r \leq \delta(G) - k + 1$, then $(r-k)(\delta(G)-r-k+1) \geq 0$. Therefore $|W| = |W_1| + |W_M| \geq \xi_{(k)}(G) + |V(H)| \geq \xi_{(k)}(G) + k + 1 > \xi_{(k)}(G^\pi)$ by Lemma 2. Thus, we may suppose $|V(H)| = r \geq \delta(G) - k + 2$, which implies $k = 3$ and $r = |V(H)| = \delta(G) - 1$. In this case we have

$$|W_1| \geq \xi_{(3)}(G) + (r - 3)(\delta(G) - (\delta(G) - 1) - 2) = \xi_{(3)}(G) - |V(H)| + 3.$$

Then, taking into account Lemma 2,

$$|W| = |W_1| + |V(H)| \geq \xi_{(3)}(G) - |V(H)| + 3 + |V(H)| \geq \xi_{(3)}(G^\pi),$$

and the theorem holds. □

Corollary 1. *Let G be a $\lambda_{(2)}$-optimal graph with $\delta(G) \geq 3$. Then*

$$\lambda_{(2)}(G^\pi) = \min\{|V(G)|, \xi(G^\pi)\}$$

for every permutation π of $V(G)$.

Proof. Since the graph is $\lambda_{(2)}$-optimal we have $\lambda_{(2)}(G) = \xi(G) \geq 2\delta(G) - 2 > \delta(G)$. Then

$$2\lambda_{(2)}(G) > \lambda_{(2)}(G) + \delta(G) = \xi(G) + \delta(G) \geq \xi(G) + 3 > \xi(G^\pi),$$

having used Lemma 2 for the last inequality. Then, as a consequence of Theorem 3.1 we have

$$\lambda_{(2)}(G^\pi) \geq \min\{|V(G)|, \xi(G^\pi)\}.$$

To end the proof it suffices to notice that $\lambda_{(2)}(G^\pi) \leq |V(G)|$, because the set of cross edges of G^π is a 2-restricted edge cut of G^π as $|V(G)| \geq 4$, and also that $\lambda_{(2)}(G^\pi) \leq \xi(G^\pi)$ follows from $\delta(G^\pi) \geq 4$, because $\delta(G^\pi) \geq 4$ clearly implies that G^π cannot be a star and has at least 4 vertices. □

Taking into account that $|V(G)| \geq \xi(G) + 2$ implies $|V(G)| \geq \xi(G^\pi)$ by means of Lemma 2, we obtain the following result as a consequence of Corollary 1.

Corollary 2. *Let G be a $\lambda_{(2)}$-optimal graph of order $|V(G)| \geq \xi(G) + 2$ and minimum degree $\delta(G) \geq 3$. Then, for every permutation π of $V(G)$, the graph G^π is $\lambda_{(2)}$-optimal.*

Corollary 3. *Let G be a $\lambda_{(3)}$-connected graph of minimum degree $\delta(G) \geq 4$. Then the following assertions hold for any permutation graph G^π.*

(i) *If $|V(G)| \geq \xi(G) + 2$ and $\lambda_{(3)}(G) \geq \xi(G) - \delta(G) + 2$, then $\lambda_{(3)}(G^\pi) \geq \xi(G^\pi)$.*
(ii) *If $|V(G)| \geq \xi(G) + 3$ and $\lambda_{(3)}(G) \geq \xi(G) - \delta(G) + 3$, then G^π is super restricted edge connected.*
(iii) *If $|V(G)| \geq \xi_{(3)}(G) + 3$ and $\lambda_{(3)}(G) \geq \xi_{(3)}(G) - \delta(G) + 3$, then $\lambda_{(3)}(G^\pi) = \xi_{(3)}(G^\pi)$.*

Proof. We prove (ii) because (i) and (iii) are similar. By Theorem 3.1 we have

$$\lambda_{(3)}(G^\pi) \geq \min\{|V(G)|, 2\lambda_{(3)}(G), \lambda_{(3)}(G) + \delta(G), \xi_{(3)}(G^\pi)\},$$

and by Lemma 2,

$$\xi_{(3)}(G^\pi) \geq \min\{\delta(G) + \xi(G) + 1, \xi_{(3)}(G) + 3\}.$$

Using Lemma 1 we have

$$\xi_{(3)}(G) + 3 \geq \xi(G) + \delta(G) - 1.$$

Thus

$$\xi_{(3)}(G^{\pi}) \geq \xi(G) + \delta(G) - 1.$$

The hypotheses imply

$$\begin{aligned} 2\lambda_{(3)}(G) &= \lambda_{(3)}(G) + \lambda_{(3)}(G) \\ &\geq \lambda_{(3)}(G) + \xi(G) - \delta(G) + 3 \\ &\geq \lambda_{(3)}(G) + 2\delta(G) - 2 - \delta(G) + 3 \\ &= \lambda_{(3)}(G) + \delta(G) + 1, \end{aligned}$$

since $\xi(G) \geq 2\delta(G) - 2$. Applying again the hypotheses,

$$\lambda_{(3)}(G) + \delta(G) + 1 \geq \xi(G) - \delta(G) + 3 + \delta(G) + 1 = \xi(G) + 4 > \xi(G) + 3.$$

Therefore

$$\begin{aligned} \lambda_{(3)}(G^{\pi}) &\geq \min\{|V(G)|, 2\lambda_{(3)}(G), \lambda_{(3)}(G) + \delta(G), \xi_{(3)}(G^{\pi})\} \\ &\geq \min\{\xi(G) + 3, \xi(G) + \delta(G) - 1\} \\ &\geq \xi(G) + 3 > \xi(G^{\pi}), \end{aligned}$$

because $\delta(G) \geq 4$. Hence by Theorem 2.3 G^{π} is super restricted edge connected. □

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