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On the acyclic disconnection and the girth[☆]

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ABSTRACT

The *acyclic disconnection*, $\overrightarrow{\omega}(D)$, of a digraph D is the maximum number of connected components of the underlying graph of $D - A(D^*)$, where D^* is an acyclic subdigraph of D. We prove that $\overrightarrow{\omega}(D) \geq g - 1$ for every strongly connected digraph with girth $g \geq 4$, and we show that $\overrightarrow{\omega}(D) = g - 1$ if and only if $D \cong C_g$ for $g \geq 5$. We also characterize the digraphs that satisfy $\overrightarrow{\omega}(D) = g - 1$, for $g = 4$ in certain classes of digraphs. Finally, we define a family of bipartite tournaments based on projective planes and we prove that their acyclic disconnection is equal to 3. Then, these bipartite tournaments are counterexamples of the conjecture $\overrightarrow{\omega}(T) = 3$ if and only if $T \cong \overrightarrow{C}_4$ posed for bipartite tournaments by Figueroa et al. (2012).

1. Introduction

The *acyclic disconnection* of a digraph was defined by Neumann-Lara in [13] as the maximum number of connected components of the underlying graph of $D - A(D^*)$ where D^* is an acyclic subdigraph of D. This definition is equivalent to other definitions in terms of vertex colorings, cycle transversals or certain subdigraphs [13]. The acyclic disconnection measure somehow the complexity of cyclic patterns of a digraph. Roughly speaking, a small value of $\overrightarrow{\omega}(D)$ implies a complex pattern of cycles in D.

The acyclic disconnection of a digraph has mainly been studied in different classes of digraphs: circulant tournaments [9,10,12], bipartite tournaments [6] and other special tournaments [11]. In [13] upper bounds were established for the acyclic disconnection in terms of invariants such as the dichromatic number (introduced by Neumann-Lara in 1982), the maximum order of an acyclic subset of vertices, $\overrightarrow{\beta}(D)$, or the number of vertices of the digraph D. The aim of this paper is to study the acyclic disconnection and its relation to the girth and the semigirth of the digraph. The directed girth is the length of a shortest cycle. An important difference between the girth in graphs and the girth in digraphs is that, for any two vertices u, v on a shortest cycle in a graph it follows that $dist(u, v) \leq g/2$, but in a digraph there are vertices on a shortest cycle such that $dist(u, v) = g - 1$. Fábrega and Fiol introduced in [5,7] the semigirth ℓ of a digraph. This parameter is related to the path structure of the digraph and plays a role similar (and is tightly related) to the girth of a graph. The semigirth ℓ has been widely used to study connectivity and some other structural properties of digraphs [1–3,5,7,8].

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In Section 3, we study the relation between the girth and the acyclic disconnection. We give a lower bound of $\overrightarrow{\omega}(D)$ in terms of the girth g and we characterize the digraphs that attain this lower bound for $g \geq 5$. The case $g = 4$ is discussed in Section 4. Under certain conditions on the digraph D, such as the order of acyclic subsets of vertices, the distance or structural conditions, we prove that $\overrightarrow{\omega}(D) \geq 4$. Moreover, we show that the characterization for $g \geq 5$ (Theorem 7) is also valid for particular classes of digraphs with girth 4, but not in general. If a bipartite tournament has a cycle, clearly its girth is 4. It was recently conjectured by Figueroa et al. [6] that a bipartite tournament T has acyclic disconnection equal to 3 if and only if T is the cycle with 4 vertices. We show a family of digraphs based on projective planes that are counterexamples to this conjecture.

2. Definitions and known results

For terminology and other general concepts, see [4]. In this paper we consider only oriented simple graphs. If D is bipartite we will write $V(D) = U_0 \cup U_1$, where U_0 and U_1 denote the partite vertex sets. For a set $X \subseteq V(D)$, we denote by $D[X]$ the subdigraph of D induced by X. We use $d^+(v)$ and $d^-(v)$ for the out-degree and the in-degree of v, respectively. A vertex v is a *source* (resp. *sink*) of D if $d^-(v) = 0$ (resp. $d^+(v) = 0$). The sequence $P = v_1v_2 \cdots v_n$ of vertices of D is a path if $v_iv_{i+1} \in A(D)$ for every $1 \leq i \leq n-1$ and $v_i \neq v_j$ for all $i \neq j$. If the vertices $v_1, v_2, \ldots, v_{n-1}$ are distinct and $v_1 = v_n$, then P is a cycle. Sometimes we denote the path P as a (v_1, v_n)-path. A k-cycle is a cycle of length k. The *girth*, denoted by g, is the length of a cycle of minimum length. A digraph D without cycles is an *acyclic digraph*. Every acyclic digraph has an acyclic ordering of its vertices, where an *acyclic ordering* $(v_1, v_2, \ldots, v_n)$ of $V(D)$ means that for every arc v_iv_j in D, we have $i < j$. An acyclic digraph has at least one sink and at least one source. A tournament T is a digraph such that there is exactly one arc between any two vertices $u, v \in V(T)$. An acyclic tournament (i.e. transitive tournament) on k vertices is denoted as TT_k. It is well known that an acyclic tournament has a unique acyclic ordering.

Let Γ_s denote the set of colors $\{c_1, c_2, \ldots, c_s\}$. Let D be a digraph and $\varphi : V(D) \to \Gamma_s$ a vertex coloring of D. The color c_α is a singular class of φ if there is $u \in V(D)$ such that $\varphi(u) = c_\alpha$ and $\varphi(v) \neq c_\alpha$ for every $v \in V(D) \setminus \{u\}$. We say that a subdigraph H of D is *proper colored* if $\varphi(u) \neq \varphi(v)$ for any two vertices $u, v \in V(H)$ such that $uv \in A(D)$. So, a proper (colored) cycle is a cycle such that any two consecutive vertices u, v on the cycle have different color. Using this terminology, the acyclic disconnection can be defined as the maximum number of colors of a vertex coloring without proper cycles. A subdigraph H is proper colored if every arc of H is a bi-colored arc of the coloring φ, where the set of *bi-colored* arcs is $\{uv \in A(D) : \varphi(u) \neq \varphi(v)\}$.

If D is not a strongly connected oriented graph, then the acyclic disconnection is the sum of the acyclic disconnection of its strongly connected components. So, in this paper we consider only strongly connected oriented graphs.

In [13] some upper bounds were established in terms of the maximum order of an acyclic induced subdigraph of D denoted by $\overrightarrow{\beta}(D)$, the dichromatic number and other invariants.

Theorem 1 (*Theorem 5.1, [13]*)**.** *Let D be a digraph. Then $\overrightarrow{\omega}(D) \leq \overrightarrow{\beta}(D)$.*

The bound of Theorem 1 is tight for oriented cycles.

3. Girth and acyclic disconnection

We study the relation between the girth and the acyclic disconnection and finally, we discuss the case $\overrightarrow{\omega}(D) = \overrightarrow{\beta}(D)$.

Theorem 2. *Every digraph D with girth $g \geq 4$ that contains a subdigraph isomorphic to an acyclic tournament of order k has $\overrightarrow{\omega}(D) \geq k + g - 3$.*

Proof. Note that $k \geq 2$. Let $(v_1, v_2, \ldots, v_k)$ be the ordering of the vertices of an acyclic sub tournament TT_k of D. Since the girth is g and D is strongly connected, the digraph contains a vertex set $U = \{v_1, v_2, \ldots, v_{k+g-4}\}$ such that $v_kv_{k+1} \cdots v_{k+g-4}$ is a path of length $g-4$. Observe that $D[U]$ is acyclic because $\{v_1, v_2, \ldots, v_k\}$ induces a TT_k and the girth is g. Clearly, $V(D) \setminus U \neq \emptyset$. Let $\varphi : V(D) \to \Gamma_{k+g-3}$ be the vertex coloring defined by

$$\varphi(x) = \begin{cases} c_i & \text{if } x = v_i \in U; \\ c & \text{if } x \notin U. \end{cases}$$

That is, every $v_i \in U$ is a singular class of color c_i and $V(D) \setminus U$ is monochromatic of color c. In order to prove that $\overrightarrow{\omega}(D) \geq k+g-3$, we suppose for a contradiction, that there exists a proper colored cycle C by φ. Since $g \geq 4$, the cycle C has at least two vertices with color different from the color c. Therefore, $|U \cap V(C)| \geq 2$. Let μ be the greatest integer such that $v_\mu \in U \cap V(C)$. Let $x, y \in V(C)$ be such that $v_\mu x, xy \in A(C)$. Then $x \notin U$ by the choice of μ and the fact that $D[U]$ is acyclic. So, $\varphi(x) = c$ and $\varphi(y) \neq \varphi(x) = c$ yielding that $y = v_\alpha$ for some α such that $1 \leq \alpha \leq \mu - 1$, because c_μ is a singular chromatic class. Hence, $C' = v_\alpha \cdots v_\mu x v_\alpha$ is a cycle of D. If $\mu \leq k$, then $v_\alpha v_\mu$ is an arc of TT_k and C' is a triangle, which is a contradiction because $g \geq 4$. So $\mu \geq k+1$. If $\alpha \geq k$ then C' has length at most $\mu - \alpha + 2 \leq k + g - 4 - k + 2 = g - 2$, which is a contradiction. Thus, $\alpha \leq k-1$ and $C' = v_\alpha v_k \cdots v_\mu x v_\alpha$ has length at most $g - 3 + 2 = g - 1$, which is again a contradiction. Hence, $\overrightarrow{\omega}(D) \geq k + g - 3$. ■

Corollary 3. *Every digraph D with girth $g \geq 4$ has $\overrightarrow{\omega}(D) \geq g - 1$.*

If D has acyclic triangles, then the order of an acyclic subtournament of D is $k \geq 3$. Hence, from Theorem 2, the next result follows.

Corollary 4. *Every digraph D with girth $g \geq 4$ that contains a TT_3 has $\overrightarrow{\omega}(D) \geq g$.*

The following corollary is an immediate consequence of Theorems 1 and 2.

Corollary 5. *Let D be a strongly connected graph with girth $g \geq 4$. If D has a subdigraph isomorphic to an acyclic tournament of order k, then $k + g - 3 \leq \overrightarrow{\omega}(D) \leq \overrightarrow{\beta}(D)$.*

The family of cycles C_n with $n \geq 4$, satisfies $k = 2, g = n - 1$ and $\overrightarrow{\beta}(C_n) = n - 1$. So, both bounds are tight in this family.

In Theorem 7 we characterize the digraphs D that satisfy $\overrightarrow{\omega}(D) = g - 1$ for $g \geq 5$. The case $g = 4$ will be discussed in Section 4. The distance from u to a vertex set S, $dist(u, S)$, is $\min_{s \in S}\{dist(u, S)\}$. The distance from S to u is analogous.

Proposition 6. *Let D be a digraph with girth $g \geq 5$ and C a cycle of length g. If there is a vertex u such that $dist(u, C) = dist(C, u) = 1$, then $\overrightarrow{\omega}(D) \geq g$.*

Proof. Let $C = v_1 v_2 \cdots v_g v_1$ be a cycle of length g. Since $dist(u, C) = dist(C, u) = 1$ it follows that there exist $v_k, v_l \in V(C)$ such that $v_k u, u v_l \in A(D)$. Then $dist(v_k, v_l) \leq 2$. By Corollary 4, $\overrightarrow{\omega}(D) \geq g$ if $dist(v_k, v_l) = 1$. Thus, $dist(v_k, v_l) = 2$ and without loss of generality we may assume that $v_2 u, u v_4 \in A(D)$. Let $\varphi : V(D) \to \Gamma_g$ be the vertex coloring defined by:

$$\varphi(x) = \begin{cases} c_i & \text{if } x = v_i \text{ and } i < g; \\ c_g & \text{if } x = u; \\ c_1 & \text{otherwise.} \end{cases}$$

If $\{u, v_3\}$ is not independent, then we are done by Corollary 4. Furthermore, since $C' = v_1 v_2 u v_4 \cdots v_g v_1$ is also a shortest cycle, $\{u, v_i\}$ is independent for $i \neq 2, 4$. We suppose, for a contradiction, that there is proper colored cycle C''. Observe that both closed neighborhood $N^+[v_g]$ and $N^-[v_1]$ are monochromatic, so

$$\{v_1, v_g\} \cap V(C'') = \emptyset. \tag{1}$$

Suppose that $v_2 \in V(C'')$ and let $x, y \in V(C'')$ be such that $xy, yv_2 \in A(C'')$. In this case $\varphi(y) \neq \varphi(v_2)$. Clearly $y \neq u$ and $y \notin V(C)$ by (1), so $\varphi(y) = c_1$ and $\varphi(x) \neq \varphi(y) = c_1$, which implies that $x \in \{u, v_3, \ldots, v_{g-1}\}$. If $x = u$, then $v_2 u y v_2$ is a triangle, if $x = v_j$ with $3 \leq j \leq g - 1$, then $v_j y v_2 v_3 \cdots v_j$ is a cycle of length $j < g$, which contradicts that g is the girth. Hence, $v_2 \notin V(C'')$.

Since $g \geq 5$, the cycle C'' has at least three vertices with color different from the color c_1. Therefore, $|V(C) \cap V(C'')| \geq 2$. Let μ be the greatest integer such that $v_\mu \in V(C) \cap V(C'')$. Thus, $4 \leq \mu \leq g - 1$. Let $x, y \in V(C'')$ be such that $v_\mu x, xy \in A(C'')$. Then $x \notin V(C) \cup \{u\}$ and so, $\varphi(x) = c_1$ and $\varphi(y) \neq \varphi(x) = c_1$, which implies that $y \in V(C) \cup \{u\}$. By the choice of μ it follows that $y \in \{u, v_3, \ldots, v_{\mu-1}\}$. If $y = v_j$ with $3 \leq j \leq \mu - 1$, then $v_\mu x v_j v_{j+1} \cdots v_\mu$ is a cycle of length less than the girth, which is a contradiction. The case $y = u$ is analogous. Thus, φ is a vertex coloring without proper cycles and $\overrightarrow{\omega}(D) \geq g$. ■

Theorem 7. *Let D be a digraph with girth $g \geq 5$. Then $\overrightarrow{\omega}(D) = g - 1$ iff $D \cong C_g$.*

Proof. Clearly $\overrightarrow{\omega}(C_g) = g - 1$. Let D be a digraph with girth g and $\overrightarrow{\omega}(D) = g - 1$. Suppose, for a contradiction, that the order of D is at least $g + 1$.

Let $C = v_1 v_2 \cdots v_g v_1$ be a cycle of length g. Let $\varphi : V(D) \to \Gamma_g$ be the vertex coloring defined by:

$$\varphi(x) = \begin{cases} c_i & \text{if } x = v_i \text{ and } i < g; \\ c_1 & \text{if } x = v_g; \\ c & \text{otherwise.} \end{cases}$$

Since $\overrightarrow{\omega}(D) = g - 1$ and φ is a coloring of g colors, there exists a proper colored cycle C'. Furthermore, there exists a vertex $y \in V(C')$ such that $\varphi(y) = c$. Let $xy, yz \in A(C')$. Then $\varphi(x) \neq c \neq \varphi(z)$ and $x, z \in V(C)$ because the vertices of the cycle C are the only vertices of D with color different from c. By Proposition 6, $\overrightarrow{\omega}(D) = g$, which is a contradiction. ■

The following results give sufficient conditions on acyclic subdigraphs to guarantee that $\overrightarrow{\omega}(D) = \overrightarrow{\beta}(D)$.

Theorem 8. *Let D be a digraph and H an acyclic subdigraph of D such that if $dist(v, H) = 1$ [resp. $dist(H, v) = 1$], then $dist(H, v) > 1$ [resp. $dist(v, H) > 1$]. Then $\overrightarrow{\omega}(D) \geq |V(H)| + 1$.*

Proof. Let $\{c_v : v \in V(H)\} \cup \{c\}$ be a set of $|V(H)| + 1$ colors and let $\varphi : V(D) \to \Gamma_{|V(D)|+1}$ be the vertex coloring defined by

$$\varphi(v) = \begin{cases} c_v & \text{if } v \in V(H); \\ c & \text{if } v \in V(D) \setminus V(H). \end{cases}$$

Let us show that φ does not produce proper colored cycles.

Let C be a cycle of D. Note that there must exist $v \notin V(H)$ such $v \in V(C)$, because by hypothesis H is acyclic. Then, there exists a path xvz of length two contained in C. Either $z \notin V(H)$ and clearly $\varphi(v) = \varphi(z) = c$ or on the contrary $z \in V(H)$. By hypothesis it follows that $x \notin V(H)$, so that $\varphi(x) = \varphi(v) = c$.

Hence, $\overrightarrow{\omega}(D) \geq |V(H)| + 1$. ■

Corollary 9. *Let D be a digraph and H an acyclic subdigraph of D such that $|V(H)| = \overrightarrow{\beta}(D) - 1$ and if $dist(v, H) = 1$ [resp. $dist(H, v) = 1$], then $dist(H, v) > 1$ [resp. $dist(v, H) > 1$]. Then $\overrightarrow{\omega}(D) = \overrightarrow{\beta}(D)$.*

4. The acyclic disconnection of digraphs with girth 4

Let D be a digraph with girth 4. If D has a transitive sub tournament of order k with $k \geq 3$, then by Theorem 2, $\overrightarrow{\omega}(D) \geq 4 + k - 3 \geq 4$. In this section we show conditions on digraphs with girth 4 in order to guarantee that $\overrightarrow{\omega}(D) \geq 4$. For some particular classes of digraphs with girth 4 we will prove that $\overrightarrow{\omega}(D) = 3$ iff $D \cong C_4$.

Lemma 10. *Let D be a trianglefree digraph with girth $g = 4$. Then $\overrightarrow{\omega}(D) \geq 4$ if one of the following is fulfilled.*

(i) *There are two in-neighbors (or out-neighbors) v_1, v_2 of some vertex of D such v_1, v_2 are not contained in any 4-cycle of D.*
(ii) *There is a path of length two not contained in a cycle of length four.*
(iii) *There is a 4-cycle C and a vertex $u \in V(D) \setminus V(C)$ such that $dist(C, u) \geq 3$ or $dist(u, C) \geq 3$.*

Proof. For (i) and (ii), let $\varphi : V(D) \to \Gamma_4$ be the vertex coloring defined by

$$\varphi(x) = \begin{cases} c_i & \text{if } x = v_i \text{ and } i \leq 3; \\ c_4 & \text{if } x \notin V(C). \end{cases}$$

In both cases, it is easy to see that, the vertex coloring φ has no proper colored cycles.

Let $C = v_1v_2v_3v_4v_1$ be a cycle in D. For (iii), let $u \in V(D) \setminus V(C)$ such that $dist(C, u) \geq 3$ (resp. $dist(u, C) \geq 3$). Let $\varphi : V(D) \to \Gamma_4$ be the vertex coloring defined by

$$\varphi(x) = \begin{cases} c_i & \text{if } x = v_i \text{ and } i \leq 3; \\ c & \text{if } x = u; \\ c_1 & \text{otherwise}. \end{cases}$$

It is easy to see that the vertex coloring φ has no proper colored cycles. ■

Theorem 11. *Let D be a digraph with girth $g = 4$ having a $C_4 = v_1v_2v_3v_4v_1$ such that $|N^-(v_1) \cap N^+(v_3)| = 1$ and $|N^-(v_2) \cap N^+(v_4)| = 1$. Then $\overrightarrow{\omega}(D) = 3$ iff $D = C_4$.*

Proof. Clearly $\overrightarrow{\omega}(C_4) = 3$. Suppose, for a contradiction, that the order of D is at least 5. Observe that if D has acyclic triangles, then $\overrightarrow{\omega}(D) \geq 4$, by Corollary 4. Thus, we must assume that D has no triangle. Let $\varphi : V(D) \to \Gamma_4$ be the vertex coloring defined by

$$\varphi(x) = \begin{cases} c_i & \text{if } x = v_i \text{ and } i < 4; \\ c_1 & \text{if } x = v_4; \\ c & \text{if } x \notin V(C). \end{cases}$$

Since $\overrightarrow{\omega}(D) = 3$ and φ is a coloring of 4 colors, we can consider a proper colored cycle C'. Note that $|V(C) \cap V(C')| \geq 2$ because $V(D) \setminus V(C)$ is monochromatic. If $v_4 \in V(C')$, then there exist two vertices $x, y \in V(C')$ such that $v_4x, xy \in A(C')$. It follows that $x \notin V(C)$, so, $\varphi(x) = c$ and $\varphi(y) \neq \varphi(x) = c$, then $y \in V(C)$. The girth is 4, so $y \neq v_3$, D has no triangle, so $y \neq v_1$ and if $y = v_2$ then $|N^-(v_2) \cap N^+(v_4)| \geq 2$, which is a contradiction with the hypothesis. Using the hypothesis $|N^-(v_1) \cap N^+(v_3)| = 1$ we obtain $v_1 \notin V(C')$. Therefore, $V(C) \cap V(C') = \{v_2, v_3\}$. Furthermore, there are two vertices $x, y \in V(C')$ such that $C' = xv_2yv_3x$, yielding that C' contains the TT_3 induced by $\{v_2, y, v_3\}$, which is a contradiction. ■

Corollary 12. *Let D be a digraph with girth $g = 4$ such that there exists a 4-cycle C with the property that $dist(C, u) \geq 2$ or $dist(u, C) \geq 2$ for all $u \in V(D) \setminus V(C)$. Then $\overrightarrow{\omega}(D) \geq 4$.*

Definition 13. Let D be a digraph with diameter $diam(D)$. The semigirth $\ell = \ell(D)$, $1 \leq \ell \leq diam(D)$, is defined as the greatest integer so that, for any two vertices u, v,

(a) if $dist(u, v) < \ell$, the shortest (u, v)-path is unique and there are no (u, v)-paths of length $dist(u, v) + 1$;
(b) if $dist(u, v) = \ell$, there is only one shortest (u, v)-path.

As a consequence of Theorem 11, we obtain the following corollary.

Corollary 14. *Every digraph D different from C_4 with girth $g = 4$ and semigirth $\ell \geq 2$ has $\overrightarrow{\omega}(D) \geq 4$.*

In the *line digraph* $L(D)$ of a digraph D, each vertex represents an arc of G, that is, $V(L(D)) = \{uv : (u, v) \in A(D)\}$; and a vertex uv is adjacent to a vertex wz if and only if $v = w$ (i.e., when the arc (u, v) is adjacent to the arc (w, z) in D).

Corollary 15. *Let D be a digraph with minimum degree $\delta \geq 2$ and girth 4. Then $\overrightarrow{\omega}(L(D)) \geq 4$.*

Proof. Since $\delta \geq 2$, the digraph D is not a directed cycle. Moreover, from [5], we know that $\ell(L(D)) = \ell(D) + 1 \geq 2$. Hence, by Corollary 14 we deduce $\overrightarrow{\omega}(L(D)) \geq 4$. ∎

Lemma 10, Corollaries 12, 14 and 15 and Theorem 11 lead to necessary conditions for a digraph with girth 4 having acyclic disconnection equal to three.

Corollary 16. *Let D be a digraph such that $\overrightarrow{\omega}(D) = 3$ and $D \not\cong \overrightarrow{C}_4$. Then D must fulfill all the following conditions.*

(i) *$g = 4$, $\ell(D) = 1$ and D has no triangles (directed or acyclic).*
(ii) *There are $v_1, v_2, v_3, v_4 \in V(D)$ such that $v_1v_2v_4$ and $v_1v_3v_4$ are induced paths in D and v_2, v_3 are independent because D has no triangles.*
(iii) *For all two in-neighbors and any two out-neighbors of some vertex of D are contained in a 4-cycle of D.*
(iv) *Every arc and every path of length two is contained in a 4-cycle.*
(v) *For every 4-cycle C and every vertex $u \in V(D) \setminus V(C)$ we have $dist(C, u) \leq 2$ and $dist(u, C) \leq 2$.*
(vi) *For every 4-cycle $C = v_1v_2v_3v_4v_1$ there are two vertices $u, w \in V(D) \setminus V(C)$ such that $v_1uv_3wv_1$ is a 4-cycle.*

4.1. Bipartite tournaments

Strongly connected bipartite tournaments are digraphs with girth 4. The acyclic disconnection of bipartite tournaments was studied in [6].

Lemma 17 (*Proposition 12 [6]*). *Every bipartite tournament T of order $n \geq 3$ has $\overrightarrow{\omega}(T) \geq 3$.*

We show a family of bipartite tournaments, and prove that the acyclic disconnection of every digraph of this family is equal to three. Thus, we will obtain a counterexample to the following conjecture posed in [6].

Conjecture 18 (*[6]*). *Let T be a bipartite tournament. Then $\overrightarrow{\omega}(T) = 3$ if and only if $T \cong \overrightarrow{C}_4$.*

Moreover, this counterexample shows that the characterization in Theorem 7 does not hold for $g = 4$. To do that, we need to recall the notion of a projective plane. A *projective plane* $(P, \mathcal{L})$ consists of a finite set P of elements called *points*, and a finite family $\mathcal{L}$ of subsets of P called *lines*, which satisfy the following conditions:

(i) Any two lines intersect at a single point.
(ii) Any two points belong to a single line.
(iii) There are four points, of which no three belong to the same line.

Definition 19. Let $\Pi = (P, \mathcal{L})$ be a projective plane of order k. We define the bipartite tournament $D_k(\Pi)$ with partite sets P and $\mathcal{L}$. And the arcs are defined as follows: For all $p \in P$ and for all $L \in \mathcal{L}$, $p \in N^+(L)$ iff p belongs to L; $L \in N^+(p)$ iff p does not belong to L.

In a projective plane $\Pi = (P, \mathcal{L})$ of order k we have $|P| = |\mathcal{L}| = k^2 + k + 1$, every $p \in P$ belongs to exactly $k + 1$ lines and every $L \in \mathcal{L}$ contains exactly $k + 1$ points. Then every $p \in P$ has out-degree in $D_k(\Pi)$ equal to $|\mathcal{L}| - (k + 1) = k^2$ and every $L \in \mathcal{L}$ has out degree equal to $k + 1$.

Remark 20. Let $\Pi = (P, \mathcal{L})$ be a projective plane of order k and $D_k(\Pi)$ the bipartite tournament given in Definition 19. Any two vertices $u, v \in P$ are contained in a 4-cycle of $D_k(\Pi)$ because by the properties of Π there exist L, L' such that $u \in N^+(L)$, $v \notin N^+(L)$, $v \in N^+(L')$, and $u \notin N^+(L')$. Analogously, any two vertices $u, v \in \mathcal{L}$ are contained in a 4-cycle of $D_k(\Pi)$.

Next, we prove that the acyclic disconnection of $D_k(\Pi)$ is equal to 3, that is, $D_k(\Pi)$ is a counterexample to Conjecture 18. We need the following lemma.

Lemma 21. *Let T be a bipartite tournament with partite sets U and V such that for every pair $u \neq v \in U$ (resp. $u \neq v \in V$), there exists a 4-cycle containing u and v. Let $\overrightarrow{\omega}(T) \geq s$ and $\varphi : V(T) \to \Gamma_s$ be a vertex coloring of T without proper colored cycles. Then $|\varphi(U)|, |\varphi(V)| \geq s - 1$. Moreover, each element c_i of $\varphi(U) \setminus \varphi(V)$ is a singular class of φ.*

Proof. Suppose, for a contradiction, that $|\varphi(V)| < s - 1$. Then there exist $u, v \in V(U)$ such that $\varphi(u), \varphi(v) \notin \varphi(V)$. By hypothesis, there is a 4-cycle containing the vertices u, v and this cycle is a proper cycle, a contradiction. Analogously each element c_i of $\varphi(U) \setminus \varphi(V)$ is a singular class of φ. ∎

Following the notation of [6], $H_\varphi(D)$ denotes the spanning subdigraph of D induced by the bi-colored arcs of φ. Clearly, the digraph induced by the bi-colored arcs is acyclic. Hence, the digraph $H_\varphi(D)$ has a source and a sink.

Theorem 22. *The bipartite tournament of a projective plane has acyclic disconnection equal to* 3.

Proof. Let Π be a projective plane and $D = D_k(\Pi)$. By Remark 20, every two vertices $u, v \in P$ or $u, v \in \mathcal{L}$ are in a 4-cycle. By Lemma 17, $\overrightarrow{\omega}(D) \geq 3$. Suppose, for a contradiction, that $\overrightarrow{\omega}(D) \geq 4$, then from Lemma 21 it follows that $|\varphi(P)|, |\varphi(\mathcal{L})| \geq 3$ for every vertex coloring φ without proper colored cycles. Let $\varphi : V(D) \to \Gamma_4$ be a vertex coloring without proper colored cycles, that is $H_\varphi(D)$ is acyclic.

Suppose that there is a sink $p_0 \in P$, that is $N^+[p_0]$ is monochromatic. Then, we can assume $\varphi(N^+[p_0]) = c_1$. So, every line not containing p_0 has color c_1. Since $|\varphi(P)| \geq 3$, let p, p' be such that $\varphi(p) \neq c_1 \neq \varphi(p')$. Let $L, L' \in N^+[p_0]$ be such that $p \in N^+(L)$ and $p' \in N^+(L')$. If $p' \notin N^+(L)$, then $LpL'p'L$ is a proper cycle, which is a contradiction. Hence, $p' \in N^+(L)$. Let $L'' \in N^+[p_0] - L$ be such that $p \in N^+(L'')$, so $p' \notin N^+(L'')$. Then, $L''pL'p'L''$ is a proper cycle, which is again a contradiction. Hence, $H_\varphi(D)$ has no sink in the set P, which yields that the sinks of $H_\varphi(D)$ are in $\mathcal{L}$.

Let L_0 be a sink of $H_\varphi(D)$. In this case $N^+[L_0]$ is monochromatic. Suppose that $\varphi(N^+[L_0]) = c_1$. There are at least two different lines $L, L' \in \mathcal{L}$ such that $\varphi(L) \neq c_1 \neq \varphi(L')$. If $N^+(L) \cap N^+(L') \cap N^+(L_0) = \emptyset$, then $LpL'p'L$ is a proper cycle for $p \in N^+(L_0) \cap N^+(L)$ and $p' \in N^+(L'_0) \cap N^+(L)$, which is a contradiction. So, $N^+(L) \cap N^+(L') \cap N^+(L_0) \neq \emptyset$. Let $\{p_0\} = N^+(L) \cap N^+(L') \cap N^+(L_0)$. Hence, if $\varphi(L_1) \neq c_1$ then $p_0 \in N^+(L_1)$. Therefore, $N^+[p_0]$ is monochromatic, that is, p_0 is a sink of $H_\varphi(D)$, a contradiction.

Since in either case there is a contradiction, $\overrightarrow{\omega}(D) = 3$. ■

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