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# Structural properties of CKI-digraphs*

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**Abstract**

A kernel of a digraph is a set of vertices which is both independent and absorbant. Let $D$ be a digraph such that every proper induced subdigraph has a kernel. If $D$ has a kernel, then $D$ is kernel perfect, otherwise $D$ is critical kernel-imperfect (for short CKI-digraph). In this work we prove that if a CKI-digraph $D$ is not 2-arc connected, then $D - a$ is kernel perfect for any bridge $a$ of $D$. If $D$ has no kernel but for all vertex $x$, $D - x$ has a kernel, then $D$ is called kernel critical. We give conditions on a kernel critical digraph $D$ so that for all $x \in V(D)$ the kernel of $D - x$ has at least two vertices. Concerning asymmetric digraphs, we show that every vertex $u$ of an asymmetric CKI-digraph $D$ on $n \geq 5$ vertices satisfies $d^+(u) + d^-(u) \leq n - 3$ and $d^+(u), d^-(u) \leq n - 5$. As a consequence, we establish that there are exactly four asymmetric CKI-digraphs on $n \leq 7$ vertices. Furthermore, we study the maximum order of a subtournament contained in a not necessarily asymmetric CKI-digraph.

**Keywords:** digraphs, kernel, circulant digraphs, critical kernel perfect.

**2010 Mathematics Subject Classification:** 05C15, 05C20.

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## 1. Introduction

A *kernel* of $D$ is a subset $K \subset V(D)$ which is independent and absorbant [18]. For undirected graphs the corresponding concept is known as *independent dominating set*. This notion of domination in graphs has received extensive attention, see [18, 19]. In terms of applications, some important questions of Facility Location, Assignment Problems, etc, are very much related to the idea of domination or independent domination on digraphs. Furthermore, the notion of kernels (or independent dominating set) has many applications and several relations to other areas, most notably to game theory [3, 12, 13, 21] and logic [8, 20]. An interesting survey of kernels in digraphs can be found in [10] and also see chapter 15 pages 401-437 of [18].

Let $D$ be a digraph such that every proper induced subdigraph has a kernel. Then $D$ is *kernel perfect* if $D$ has a kernel, otherwise $D$ is *critical kernel imperfect* (for short CKI or CKI-digraph). For instance, a directed cycle of odd length has no kernel, but a directed cycle of even length is kernel perfect. Kernel perfect digraphs have been extensively studied because their relationship with perfect graphs [1, 5, 6, 9]. Recently Galeana-Sánchez [14] has given a new characterization of perfect graphs using asymmetric kernel perfect digraphs. However, CKI-digraphs have been less studied. There are operations on digraphs that preserve the property of being kernel perfect or CKI. Duchet and Meyniel [11] prove that splitting a vertex of a digraph $D$ and then subdividing the resulting arc, these properties are preserved. Moreover, they give another operation that respect these properties, which roughly speaking consists of replacing an arc $a$ by a directed path of length 3, whenever $D-a$ has a kernel or be kernel perfect, respectively. Also these authors point out that changing the directions of every arc of $D$ is not such an operation.

Berge and Duchet [6] proved that a CKI-digraph is strongly connected. A digraph $D$ is said to be *strongly connected* (or connected) if for any pair of vertices $x, y \in V(D)$ there exists a path from $x$ to $y$. An *arc cut* of $D$ is a subset of arcs $S$ such that $D - S$ is not strongly connected. The *arc connectivity*, $\lambda(D)$, is the smallest cardinality of an arc cut. It is well known [17] that for any digraph $D$, $\lambda(D) \leq \delta(D)$. In this paper, we prove that if $D$ is a CKI-digraph and $\lambda(D) = 1$, then $D - a$ is a kernel perfect digraph for any bridge $a$ of $D$.

A digraph $D$ with no kernel is *kernel critical* if $D - x$ has a kernel for every $x \in V(D)$. Note that a CKI-digraph is a kernel critical digraph but there are kernel critical digraphs that are not CKI [18]. We find sufficient conditions on kernel critical digraphs such that for every $x \in V(D)$ the kernel of $D - x$ has at least two vertices. Then we focus on asymmetric CKI-digraphs. We show that every vertex $u$ of an asymmetric CKI-digraph $D$ on $n \geq 5$ vertices satisfies $d^+(u) + d^-(u) \leq n - 3$ which clearly implies that if $D$ is $d$-regular, then $d \leq (n-3)/2$. Moreover, we state that $d^+(u), d^-(u) \leq n-5$. These results allow us to establish that every asymmetric CKI-digraph is a $\vec{C}_3$, a $\vec{C}_5$, a $\vec{C}_7$, a $\vec{C}_7(1,2)$ or has $n \geq 8$ vertices. More characterization results of asymmetric CKI can be founded in [16]. Finally, for CKI-digraphs $D$ not necessarily asymmetric on $n \geq 4$ vertices we study the maximum order of a subtournament contained in $D$. We establish that if we remove one or two vertices from a CKI-digraph, or any independent set of vertices, the resulting digraph is not a tournament.

### 1.1. Notation and known results

For general terminology and definitions see [2, 4].

A digraph is a finite nonempty set of vertices $V(D)$ and a set $A(D)$ of ordered pairs of distinct vertices $(x, y)$ called arcs. The set $N^+(x) = \{y \in V(D) : (x, y) \in A(D)\}$ (resp. $N^-(x) = \{y \in V(D) : (y, x) \in A(D)\}$) is called the *out-neighborhood* (resp. *in-neighborhood*) of $x$. The *out-degree* of $x$ is $d^+(x) = |N^+(x)|$ and the *in-degree* of $x$ is $d^-(x) = |N^-(x)|$. The *maximum out-degree* is denoted by $\Delta^+(D)$ and the *maximum in-degree* is denoted by $\Delta^-(D)$. Given a subset $S \subset V(D)$ we denote by $D[S]$ the subdigraph of $D$ induced by $S$.

An arc $(u, v) \in A(D)$ is called *asymmetric* (resp. *symmetric*) if $(v, u) \notin A(D)$ (resp. $(v, u) \in A(D)$). A digraph is asymmetric, (resp. symmetric) if for every arc $(u, v) \in A(D)$, then $(v, u) \notin A(D)$ (resp. $(v, u) \in A(D)$). The spanning subdigraph induced by the set of asymmetric (symmetric) arcs of $D$ is denoted by $Asym(D)$ ($Sym(D)$). A digraph is *transitive* if $(u, v), (v, w) \in A(D)$, then $(u, w) \in A(D)$. Clearly, a transitive oriented graph is acyclic. A *tournament* is an asymmetric digraph where every pair of distinct vertices are adjacent. A tournament is transitive if and only if it is acyclic. We denote a tournament on $k$ vertices as $T_k$. Let $n$ be a positive integer and $A = \{a_1, a_2, \ldots, a_d\} \subset \mathbb{Z}_n - 0$. The *circulant digraph* $\vec{C}_n(A)$ has set of vertices the integers modulo $n$, and vertex $u$ is adjacent to the vertices $u + A = \{u + a_i \pmod{n} : a_i \in A\}$.

A set $S \subset V(D)$ is *independent* if for all $x, y \in S$, $(x, y) \notin A(D)$. A set $S \subset V(D)$ is *absorbant* if for every vertex $x \in V(D) \setminus S$ there is a vertex $y \in S$ such that $(x, y) \in A(D)$. Let $U_1, U_2$ be two subsets of vertices of $D$. An $U_1U_2$*-arc* is an arc $(u_1, u_2)$ of $D$ such that $u_1 \in U_1$ and $u_2 \in U_2$. If $U_1$ consists of a single vertex $\{u_1\}$, we simply write an $u_1U_2$*-arc*, and analogously if $U_2 = \{u_2\}$ we write an $U_1u_2$*-arc*.

In order to prove our results we need the following known results.

**Theorem 1.1.** [6] *A CKI-digraph is strongly connected.*

**Theorem 1.2.** [15] *If $D$ is a CKI-digraph, then $Asym(D)$ is strongly connected.*

**Theorem 1.3.** [15] *A digraph $D$ is kernel perfect if and only if for every strong component $\alpha$ of $Asym(D)$, $D[V(\alpha)]$ is kernel perfect.*

**Theorem 1.4.** [22] *Every digraph with no odd cycle is kernel perfect.*

## 2. Main results

Applying the above mentioned theorems we obtain the following result.

**Theorem 2.1.** *Let $D$ be a CKI-digraph and $a \in A(D)$ a bridge. Then $D - a$ is kernel perfect.*

*Proof.* By Theorem 1.1, the digraph $D$ is strongly connected. Suppose that $a$ is a bridge of $D$ and $V^-, V^+$ is a partition of $V(D)$ such that the unique $V^-V^+$-arc is the bridge $a$. Moreover, by Theorem 1.2, $Asym(D)$ is strongly connected yielding that $a$ must be an asymmetric arc. Thus $a$ is also a bridge in $Asym(D)$. Let $\alpha$ be a strongly connected component of $Asym(D-a)$, thus $\alpha \subset V^-$ or $\alpha \subset V^+$. Therefore $D[V(\alpha)]$ is kernel perfect because $D$ is CKI. By Theorem 1.3, $D - a$ is kernel perfect, so the theorem holds. □

Figure 1 shows a CKI-digraph $D$ different from an odd cycle with $\lambda(D) = 1$. We can check that $D-(0,1)$, $D-(8,0)$, $D-(7,8)$ are kernel perfect digraphs. This CKI-digraph has the property that changing the direction of its arcs the resulting digraph is kernel-perfect. It was given by Duchet and Meyniel [11] in order to disprove a conjecture of Chvátal and Berge claiming that $D$ is kernel-perfect if and only if its reverse $D^{-1}$ is kernel perfect.

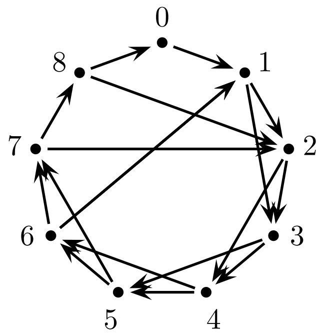


Figure 1: A CKI-digraph.

As we said in the Introduction, a digraph $D$ with no kernel is kernel critical if $D - v$ has a kernel for every $v \in V(D)$. Let $K_v$ be a kernel of $D-v$ for a given vertex $v$. Clearly, there is no $vK_v$-arc, because otherwise $K_v$ would be a kernel of $D$. Also there is a $K_vv$-arc, because otherwise $K_v \cup \{v\}$ would be a kernel of $D$. Therefore we can write the following lemma.

**Lemma 2.2.** *Let $D$ be kernel critical and let $v \in V(D)$. Let $K_v$ be a kernel of $D-v$. Then there is no $vK_v$-arc and there is a $K_vv$-arc.*

Let $D$ be a kernel critical digraph and $x \in V(D)$. The following theorem provides conditions on $D$ so that the cardinality of a kernel of $D - x$ is at least two.

**Theorem 2.3.** *Let $D$ be a kernel critical digraph. Then any kernel of $D - x$ for all $x \in V(D)$ has at least two vertices if one of the following assertions holds:*

*(i) Any directed triangle of $D$ has at least two symmetric arcs.*

*(ii) The digraph $D$ is free of directed triangles.*

*Proof.* The unique kernel critical digraph on at most three vertices is $D \cong \vec{C}_3$. But $\vec{C}_3$ does not satisfy the requirements of the theorem. So, we assume that $|V(D)| \geq 4$. We reason by contradiction supposing that there exists a vertex $x^* \in V(D)$ such that $D - x^*$ has a kernel $\{v\}$. From Lemma 2.2, it follows that $(v, x^*) \in A(Asym(D))$. Let us consider a kernel $K_v$ of $D - v$. By Lemma 2.2, $x^* \notin K_v$. Let $h \in K_v$ be such that $(x^*, h) \in A(D)$. Again by Lemma 2.2, $(v, h) \notin A(D)$. As $\{v\}$ is a kernel of $D - x^*$ and $h \neq x^*$, then $(h, v) \in A(Asym(D))$. Therefore, the directed triangle $(v, x^*, h, v)$ in $D$ has at most one symmetric arc. This is a contradiction to item $(i)$ and clearly to item $(ii)$. □

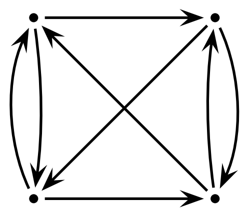

Figure 2: The CKI-digraph $\vec{C}_4(1, 2)$.

The hypothesis of Theorem 2.3 can not be eliminated as shown by the CKI-digraph $\vec{C}_4(1, 2)$ depicted in Figure 2. This digraph has directed triangles with only one symmetric arc and we can check that for all vertex $x$ of $\vec{C}_4(1, 2)$, $\vec{C}_4(1, 2) - x$ has a kernel of just one vertex.

### 2.1. Results for asymmetric CKI-digraphs

Next, we deal with asymmetric CKI-digraphs. The following result holds for every strongly connected asymmetric digraph.

**Lemma 2.4.** *Let $D$ be an strongly connected asymmetric $\vec{C}_3$-free digraph on $n \geq 5$ vertices. Then $d^-(u) + d^+(u) \leq n - 2$ for every vertex $u$.*

*Proof.* Suppose that there is a vertex $u \in V(D)$ such that $d^-(u) + d^+(u) = n - 1$. Since $D$ is strongly connected, both $N^+(u)$ and $N^-(u)$ are non-empty. Moreover, there exists an arc $(x, y)$ with $x \in N^+(u)$ and $y \in N^-(u)$ because $D$ is asymmetric and strongly connected. Then $(u, x, y, u)$ is an induced $\vec{C}_3$ contradicting that $D$ is $\vec{C}_3$-free. Therefore $d^-(u) + d^+(u) \leq n - 2$ for all vertex $u \in V(D)$. □

**Theorem 2.5.** *Let $D$ be an asymmetric CKI-digraph on $n \geq 5$ vertices. The following assertions hold:*

*(i)* *For all vertex $u$, $d^-(u) + d^+(u) \leq n - 3$.*

*(ii)* *The number of arcs of $D$ is at most $n(n-3)/2$.*

*(iii)* *Let $T_n$ be a tournament on $n$ vertices and let $M$ be a set of arcs of $T_n$. If $D$ is isomorphic to $T_n - M$ then $|M| \geq n$.*

*Proof.* $(i)$ Note that $D$ is $\vec{C}_3$-free because it is CKI. By Lemma 2.4, $d^-(u)+d^+(u) \leq n-2$ for all $u \in V(D)$. Thus, we reason by contradiction supposing that there exists $u \in V(D)$ such that $d^-(u) + d^+(u) = n - 2$. Then there exists a unique $z \in V(D)$ such that $\{u, z\}$ is an independent set and $N^+(u) \cup N^-(u) = V(D) \setminus \{u, z\}$. Since $D$ is $\vec{C}_3$-free, there is no $N^+(u)N^-(u)$-arc. Denote by $X = N^+(u) \cap N^-(z)$ and observe that $X$ is non-empty because $D$ is strongly connected. Note also that $X \neq N^+(u)$ because otherwise $N^+(u) \subseteq N^-(z)$ and $\{u, z\}$ is a kernel of $D$, which is a contradiction. Then $N^+(u) \setminus X \neq \emptyset$. Observe that there is $w_0 \in N^+(u) \setminus X$ such that $N^+(w_0) \cap X \neq \emptyset$, otherwise there is no path from $N^+(u) \setminus X$ to $X$. Hence, $\{w_0, z\}$ is an independent set because otherwise for $x \in N^+(w_0) \cap X$, $(z, w_0, x, z)$ is a $\vec{C}_3$ in $D$ which is a contradiction. Let $K_u$ be a kernel of $D - u$. By Lemma 2.2, $K_u \cap N^+(u) = \emptyset$ and there is some $K_u u$-arc. Hence, $K_u \subset N^-(u) \cup \{z\}$. Since $\{w_0, z\}$ is an independent set it follows that there exists a vertex $y \in N^-(u) \cap K_u$ such that $(w_0, y) \in A(D)$. This is a contradiction because there is no $N^+(u)N^-(u)$-arc as $D$ is $\vec{C}_3$-free. Therefore $d^-(u) + d^+(u) \leq n - 3$ for all $u \in V(D)$.

$(ii)$ This results is clear because $2|A(D)| = \sum_{u \in V(D)}(d^-(u) + d^+(u)) \leq n(n-3)$.

$(iii)$ Suppose that a CKI-digraph $D$ is isomorphic to $T_n - M$ where $M$ is a set of arcs. Then $|A(D)| = \binom{n}{2} - |M| \leq n(n-3)/2$. Therefore $|M| \geq n$. $\square$

**Remark 2.6.** *The upper bound on the number of arcs given in Theorem 2.5, is attained for $n = 5$ and $\vec{C}_5$ is the extremal graph. Also it is attained for $n = 7$ and the circulant digraph $\vec{C}_7(1,2)$ of Figure 3, is an extremal graph.*

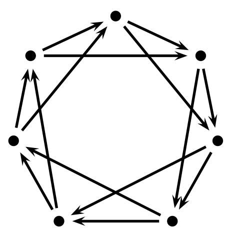

Figure 3: An asymmetric CKI-digraph on 7 vertices and 14 arcs.

**Corollary 2.7.** *Let $D$ be an asymmetric $d$-regular CKI-digraph on $n \geq 5$ vertices. Then $d \leq (n-3)/2$.*

*Proof.* By Theorem 2.5 (*ii*) we have $|A(D)| = \sum_{u \in V(D)} d^+(u) = nd \leq n(n-3)/2$. Then $d \leq (n-3)/2$. □

**Lemma 2.8.** *Let $D$ be an asymmetric CKI-digraph and $x \in V(D)$ such that $N^+(x) = \{y\}$. Then $\{w, y\}$ is an independent set of $D$ for all $w \in N^-(x)$.*

*Proof.* Since $D$ is $\vec{C}_3$-free, there is no $yN^-(x)$-arc in $D$. If $(w, y) \in A(D)$ for some $w \in N^-(x)$, then any kernel $K_w$ of $D - w$ satisfies that $K_w \cap N^+(w) = \emptyset$, so $x, y \notin K_w$. Hence $x$ is not absorbed for any element of $K_w$ which is a contradiction. Therefore $\{w, y\}$ is an independent set for all $w \in N^-(x)$. □

**Theorem 2.9.** *For every asymmetric CKI-digraph $D$ on $n \geq 6$ vertices the maximum out-degree is $\Delta^+ \leq n-5$ and the maximum in-degree is $\Delta^- \leq n-5$.*

*Proof.* Let $D$ be a CKI-digraph. By Theorem 2.5 (*i*), $d^-(v) + d^+(v) \leq n-3$, and by Theorem 1.1, $d^-(v), d^+(v) \geq 1$ for all $v \in V(D)$. Therefore it follows that $d^+(v), d^-(v) \leq n-4$ for all $v \in V(D)$. First let us see that the maximum in-degree is at most $n-5$. We reason by contradiction supposing that there is a vertex $v$ such that $d^-(v) = n-4$, so $N^+(v) = \{w\}$. Hence $\{a, b\} = V(D) \setminus (N^-(v) \cup \{v, w\})$. Since $D$ is CKI it follows that $D$ is $\vec{C}_3$-free. Therefore $N^+(w) \subseteq \{a, b\}$. W.l.g. suppose that $a \in N^+(w)$. We have $(b, a) \notin A(D)$, because otherwise $\{v, a\}$ would be a kernel of $D$; and $(a, b) \in A(D)$, because otherwise $\{v, a, b\}$ would be a kernel of $D$. Then, $(b, w) \notin A(D)$ because $D$ is $\vec{C}_3$-free, and $(w, b) \notin A(D)$ because $\{v, b\}$ is not a kernel of $D$. Then $\{w, b\}$ is an independent set and $N^+(b) \subseteq N^-(v)$. Let $v' \in N^-(v) \cap N^+(b)$. Then $D$ contains the directed 5-cycle $C = (v', v, w, a, b, v')$ which can not be induced because $D$ is CKI and $n \geq 6$. By Lemma 2.8, $\{v', w\}$ is an independent set for all $v' \in N^-(v)$. Then the only possible arc in $D$ is $(a, v')$. Thus assume $(a, v') \in A(D)$. Let $K_a$ be a kernel of $D - a$. By Lemma 2.2, $N^+(a) \cap K_a = \emptyset$. To absorb $b$ there must exist some vertex $v'' \in K_a \cap N^-(v)$ such that $(b, v'') \in A(D)$. By Lemma 2.8, $\{v'', w\}$ is independent, and since $D$ is $\vec{C}_3$-free, $\{v'', a\}$ is also independent. Then $(v, w, a, b, v'', v)$ is an induced 5-cycle which is a contradiction. Therefore $d^-(v) \leq n-5$ for all $v \in V(D)$.

Finally, let us see that the maximum out-degree is at most $n-5$. We reason by contradiction supposing that there is a vertex $v$ such that $d^+(v) = n-4$ so that $N^-(v) = \{w\}$. Hence $\{a, b\} = V(D) \setminus (N^+(v) \cup \{v, w\})$. Since $D$ is $\vec{C}_3$-free, $N^-(w) \subseteq \{a, b\}$, say $(a, w) \in A(D)$. Let $K_v$ be a kernel of $D - v$. By Lemma 2.2, it follows that $N^+(v) \cap K_v = \emptyset$, yielding $K_v \subseteq \{w, a, b\}$. Thus, $|K_v| = 2$ by Theorem 2.3. By Lemma 2.2, $K_v \neq \{a, b\}$ because $\{v, a\}$ and $\{v, b\}$ are independent. Then $K_v = \{w, b\}$. Since $D$ is $\vec{C}_3$-free, $N^+(v) \subseteq N^-(b)$, yielding $d^-(b) = n-4$ which is a contradiction. Therefore $d^+(v) \leq n-5$ for all $v \in V(D)$. □

**Corollary 2.10.** *Let $D$ be an asymmetric CKI-digraph on $n \geq 6$ vertices. Then the maximum tournament contained in $D$ has at most $n-4$ vertices.*

**Remark 2.11.** *The circulant digraph $\vec{C}_7(1,2)$ depicted in Figure 3, shows that Theorem 2.9 and Corollary 2.24 are best possible at least for 7 vertices.*

In what follows we apply the above results on asymmetric CKI-digraphs of order at most 9.

**Remark 2.12.** [7] *The unique 2-regular digraph of girth 4 is the circulant digraph $\vec{C}_7(1,2)$ depicted in Figure 3.*

**Theorem 2.13.** *Every asymmetric CKI-digraph on 7 vertices is $\vec{C}_7$ or the circulant digraph $\vec{C}_7(1,2)$.*

*Proof.* By Theorem 2.9, $\Delta^-, \Delta^+ \leq 2$. Hence, if $d^+(u) = 2$ for every vertex $u$, then $D$ is 2-regular and it is $\vec{C}_7(1,2)$ by Remark 2.12. We assume that there exists a vertex $x$ such that $N^+(x) = \{y\}$. Let us show that $d^+(y) = 1$ in which case $D = \vec{C}_7$ and the theorem holds. We reason by contradiction supposing that $N^+(y) = \{y_1, y_2\}$. By Lemma 2.8, $\{x', y\}$ is an independent set for all $x' \in N^-(x)$, and $\{x, y_i\}$, $i = 1, 2$, is an independent set because $d^+(x) = 1$ and $D$ is $\vec{C}_3$-free. Also note that $x \in K_y$ for all kernel $K_y$ of $D - y$, $N^+(y) \cap K_y = \emptyset$ and by Theorem 2.3 there exists a vertex $z \in V(D) \setminus (N^-(x) \cup N^+(y) \cup \{x, y\})$ such that $\{x, z\} \subseteq K_y$.

If $N^-(x) = \{x_1, x_2\}$, then $\{z\} = V(D) \setminus (N^-(x) \cup N^+(y) \cup \{x, y\})$, $K_y = \{x, z\}$ and $(y_i, z) \in A(D)$ for $i = 1, 2$. Then $\{y, z\}$ is an independent set because $d^+(y) = 2$ and $D$ is $\vec{C}_3$-free. Hence, $N^+(z) \subseteq N^-(x)$, say $(z, x_2) \in A(D)$. Thus, $(x, y, y_i, z, x_2, x)$ is a $\vec{C}_5$ for $i = 1, 2$. Since $D$ is CKI, these cycles are not induced, so $(y_1, x_2), (y_2, x_2) \in A(D)$ yielding $d^-(x_2) \geq 3$ which is a contradiction.

Therefore $N^-(x) = \{x_1\}$ and $\{z_1, z_2\} = V(D) \setminus (N^-(x) \cup N^+(y) \cup \{x, y\})$. It follows that $K_y = \{x, z_1, z_2\}$ because if $K_y = \{x, z_i\}$ for some $i \in \{1, 2\}$, then $d^-(z_i) \geq 3$, which is a contradiction. Moreover, if $N^-(z_i) = \{y_1, y_2\}$ for some $i \in \{1, 2\}$, then $N^+(z_i) = \{x_1\}$ (because $z_1$ and $z_2$ are independent). By Lemma 2.8, $\{x_1, y_j\}$, $j = 1, 2$ is independent, yielding $(x, y, y_j, z_i, x_1, x)$ is an induced $\vec{C}_5$ for $j = 1, 2$, which is a contradiction. Therefore, we may assume that $(y_1, z_1), (y_2, z_2) \in A(D)$. Then, $\{y, z_i\}$, $i = 1, 2$ is an independent set.

Let $K_x$ be a kernel of $D - x$. By Lemma 2.2, $x_1 \in K_x$ and $y \notin K_x$, yielding $\{y_1, y_2\} \cap K_x \neq \emptyset$. Without loss of generality, suppose $y_1 \in K_x$. Then $z_1 \notin K_x$ and clearly $(z_1, x_1) \notin A(D)$ because otherwise $(x, y, y_1, z_1, x_1, x)$ is an induced $\vec{C}_5$ which is a contradiction. Hence $N^+(z_1) = \{y_2\}$ and $y_2 \in K_x$ to absorb $z_1$, that is $K_x = \{x_1, y_1, y_2\}$. Therefore $N^-(x_1) = \{z_2\}$ and $(x, y, y_2, z_2, x_1, x)$ is an induced $\vec{C}_5$ which is a contradiction. □

In the following corollary we establish that there are exactly four asymmetric CKI-digraphs on $n \leq 7$ vertices.

**Corollary 2.14.** *Every asymmetric CKI-digraph is a* $\vec{C}_3$*, a* $\vec{C}_5$*, a* $\vec{C}_7$*, a* $\vec{C}_7(1,2)$ *or has* $n \geq 8$ *vertices.*

*Proof.* Let $D$ be an asymmetric CKI-digraph on $n$ vertices. By Theorem 1.4, $D$ has an odd cycle. If $D$ contains a directed triangle, then $D = \vec{C}_3$ and we are done. Suppose that $D$ contains a cycle of length 5, say $\vec{C}_5 = (x_0, x_1, x_2, x_3, x_4, x_0)$. For $n = 5$, by Theorem 2.5, $d^-(x_i) + d^+(x_i) \leq 2$. Then by Theorem 1.1, $d^-(x_i) = d^+(x_i) = 1$ yielding $D = \vec{C}_5$ and we are done. For $n = 6$, $d^-(x_i) \geq 2$ for some $x_i$. However, from Theorem 2.9, it follows that $d^-(x_i), d^+(x_i) \leq 1$ which is a contradiction. For $n = 7$ the result follows from Theorem 2.13. Then $n \geq 8$. □

**Corollary 2.15.** *Every arc* $(u, v)$ *of an asymmetric CKI-digraph on 8 vertices satisfies* $d^-(u) + d^+(v) \leq 5$.

*Proof.* By Theorem 2.9, $\Delta^+, \Delta^- \leq 3$. Hence, the result holds if $\Delta^+ \leq 2$. So, assume that there is a vertex $v \in V(D)$ such that $N^+(v) = \{v_1, v_2, v_3\}$ and let us show that every vertex $u \in N^-(v)$ has $d^-(u) \leq 2$. So assume that $|N^-(u)| = 3$ for some $(u, v) \in A(D)$. Then $V(D) = N^-(u) \cup N^+(v) \cup \{u, v\}$ and $d^+(u) \geq 2$ because if not, every kernel $K_v$ of $D - v$ must be $K_v = \{u\}$ contradicting Theorem 2.3. Thus, we may assume that $N^+(u) = \{v, v_1\}$, by Theorem 2.5. Since $D$ is $\vec{C}_3$-free, there is no $v_1N^-(u)$-arc. Hence $N^+(v_1) \subseteq \{v_2, v_3\}$. If $\{v_2, v_3\}$ is independent, then $\{u, v_2, v_3\}$ is a kernel of $D$ because both $\{u, v_2\}$ and $\{u, v_3\}$ are independent sets, which is a contradiction. We can assume that $(v_2, v_3) \in A(D)$. If $v_3 \in N^+(v_1)$, then $\{u, v_3\}$ is a kernel of $D$ which is a contradiction. Therefore $N^+(v_1) = \{v_2\}$ and by Lemma 2.8, $\{v, v_2\}$ must be independent which is a contradiction. Therefore every vertex $u \in N^-(v)$ has $d^-(u) \leq 2$. □

**Corollary 2.16.** *Every asymmetric CKI-digraph on 8 vertices has a vertex* $u$ *such that* $d^-(u) + d^+(u) \leq 4$.

*Proof.* By Theorem 2.5, $d^-(u) + d^+(u) \leq 5$ for all $u \in V(D)$. Suppose that there is an asymmetric CKI-digraph $D$ on 8 vertices such that $d^-(u) + d^+(u) = 5$ for all $u \in V(D)$. By Theorem 2.9, $\Delta^+, \Delta^- \leq 3$. Then, by our assumption we have $\{d^-(u), d^+(u)\} = \{2, 3\}$. Since $\sum_{u \in V(D)} d^-(u) = \sum_{u \in V(D)} d^+(u)$ there are 4 vertices with out-degree 2, and 4 vertices with out-degree 3. Let $(u, v) \in A(D)$ be such that $d^+(u) = 2$ and $d^+(v) = 3$. Since $D$ is $\vec{C}_3$-free, $N^-(u) \cap N^+(v) = \emptyset$. Let $\{v_1, v_2, v_3\} = N^+(v)$ and assume $\{v_1\} = N^+(u) \cap N^+(v)$. Note that, since $D$ is $\vec{C}_3$-free, there is no $v_1N^-(u)$-arc. Hence $N^+(v_1) = \{v_2, v_3\}$. If $\{v_2, v_3\}$ is independent, then $\{u, v_2, v_3\}$ is a kernel of $D$ because both $\{u, v_2\}$ and $\{u, v_3\}$ are independent sets, which is a contradiction. We can assume that $(v_2, v_3) \in A(D)$ which yields $\{u, v_3\}$ is a kernel of $D$ which is a contradiction. Hence, $D$ does not exist. □

**Corollary 2.17.** *If* $D$ *is a* $d$*-regular asymmetric CKI-digraph on 9 vertices, then* $d \leq 2$.

*Proof.* By Corollary 2.7, $d \leq 3$. Suppose that there is an asymmetric CKI-digraph $D$ on 9 vertices such that $d^-(u) = d^+(u) = 3$ for all $u \in V(D)$. Let $(u, v) \in A(D)$. Since $D$ is $\vec{C}_3$-free, $N^-(u) \cap N^+(v) = \emptyset$. Let $\{w\} = V(D) \setminus (N^-(u) \cup N^+(v) \cup \{u, v\})$, $N^+(v) = \{v_1, v_2, v_3\}$, $N^-(u) = \{u_1, u_2, u_3\}$. Since $N^+(u) - v \subset \{w, v_1, v_2, v_3\}$ and $d^+(u) = 3$, we may assume $v_1 \in N^+(u) \cap N^+(v)$. Then there is no $v_1 N^-(u)$-arc because $D$ is $\vec{C}_3$-free. Hence $N^+(v_1) = \{v_2, v_3, w\}$ and $\{v, w\}$ is an independent set. W.l.g. let $N^-(v) = \{u, u_1, u_2\}$, then there is no $N^+(v)\{u_1, u_2\}$-arc. Reasoning as above we have $N^-(u_1) = \{u_2, u_3, w\}$. It follows that $N^-(u_2) \subseteq \{u_3, w\}$ which is a contradiction because we are assuming that $d^-(u_2) = 3$. Therefore there is no 3-regular asymmetric CKI-digraph on 9 vertices. □

The digraph $D$ depicted in Figure 1, shows that there are CKI-digraphs on 9 vertices which are not regular.

### 2.2. Results for not necessarily asymmetric CKI-digraphs

In this subsection we deal with not necessarily asymmetric CKI-digraphs. In this case, there are infinite families of CKI-digraphs of any order $n \geq 4$. Galeana-Sánchez and Neumann-Lara [15] proved that $\vec{C}_n(1, \pm 2, \ldots, \pm(s+1))$, where $s \geq 1$ and $s+2$ does not divide $n$, is a $CKI$-digraph, and clearly it is not asymmetric. Then the results of the above subsection do not hold in general, and particularly we emphasize that Corollary 2.14 does not work.

Concerning the maximum order of a tournament contained in a not necessarily asymmetric CKI-digraph of order $n$, Corollary 2.24 does not apply for the general case. Thus, in the following results we establish that if we remove one or two vertices from a CKI-digraph, or any independent set of vertices, the resulting digraph is not a tournament. With this aim we recall that a *maximal path* is a directed path that cannot be extended to a longer directed path from either beginning or ending. First, we show that if we remove a vertex $x$ from a CKI-digraph $D$, then $D - x$ is not a tournament.

**Lemma 2.18.** *Let $D$ be a CKI-digraph on at least 4 vertices and $x \in V(D)$. Then $D - x$ is not an asymmetric digraph having maximal paths of length one.*

*Proof.* We reason by contradiction assuming that there exists a vertex $x$ such that $D - x$ is an asymmetric digraph containing maximal paths of length one. Let $(x_1, x_s)$ be a maximal path of length one of $D - x$, i.e., then $d^-_{D-x}(x_1) = 0$ and $d^+_{D-x}(x_s) = 0$. Furthermore, as $D$ is CKI, $Asym(D)$ is strongly connected by Theorem 1.2. Hence, the arcs $(x, x_1), (x_s, x) \in A(Asym(D))$ and thus $(x, x_1, x_s, x)$ form an induced $\vec{C}_3$ of $D$. This contradicts that $D$ is a CKI-digraph. Thus, the theorem is proved. □

**Proposition 2.19.** *Let $D$ be a CKI-digraph on at least 4 vertices and $x \in V(D)$. Then $D - x$ is not a tournament.*

*Proof.* Suppose that there exists a vertex $x$ such that $D - x$ is a tournament which must be transitive because $D - x$ is kernel perfect. Thus $D - x$ has maximal paths of length one which is a contradiction to Lemma 2.18. Thus the result holds. $\square$

Let $D$ be a digraph with vertex set $\{v_1, v_2, \ldots, v_n\}$, and let $G_1, G_2, \ldots, G_n$ be digraphs which are pairwise vertex-disjoint. The *composition* $D[G_1, G_2, \ldots, G_n]$ is the digraph with vertex set $V(G_1) \cup V(G_2) \cup \cdots \cup V(G_n)$ and arc set $\cup_{i=1}^{n} A(G_i) \cup \{g_i g_j : g_i \in V(G_i), g_j \in V(G_j), v_i v_j \in A(D)\}$.

The following theorem generalizes Proposition 2.19.

**Theorem 2.20.** *Let $D$ be a CKI-digraph on at least 4 vertices and $H$ an independent set of vertices of $D$. Then $D - H$ is not a composition $T_k[G_1, G_2, \ldots, G_k]$, where $T_k$ is a tournament and $G_1, G_2, \ldots, G_k$ are pairwise vertex-disjoint digraphs.*

*Proof.* From Theorem 1.1, $D$ is strongly connected, and so $|V(D) \setminus H| \geq 2$. We assume by contradiction that $D - H = T_k[G_1, G_2, \ldots, G_k]$, where $T_k$ is a transitive tournament having the hamiltonian path $(v_1, \ldots, v_k)$. Note also that $G_i$ are kernel perfect, because $D - H$ is kernel perfect. As a transitive tournament is disconnected, the composition $T_k[G_1, G_2, \ldots, G_k]$ is disconnected too. Thus $H$ must be a cut set of $D$ and also a cut set of $Asym(D)$ (recall that by Theorem 1.2, $Asym(D)$ is strongly connected). As every vertex $g_k \in V(G_k)$ is disconnected from every vertex $g_1 \in V(G_1)$, there exist $h \in H$, $x \in V(G_j)$ and $y \in V(G_i)$ with $i < j$ such that $(g_k, \ldots, x, h, y, \ldots, g_1)$ is a path in $Asym(D)$. Since $D - H$ is a composition and $i < j$, it follows that $(x, h, y, x)$ is an induced $\vec{C}_3$, which is a contradiction because $D$ is CKI. $\square$

As a direct consequence of Theorem 2.20, we obtain the following corollary.

**Corollary 2.21.** *Let $D$ be a CKI-digraph and $T_k$ a tournament. Let $T_k[G_1, G_2, \ldots, G_k]$ be a composition of the same order as $D$, and $U$ an induced subdigraph of $T_k[G_1, G_2, \ldots, G_k]$. Then $D \ncong T_k[G_1, G_2, \ldots, G_k] - A(U)$.*

Note that if $G_i$ is a single vertex for every $i = 1, \ldots, k$, then $T_k[G_1, G_2, \ldots, G_k]$ is a tournament. In this case Theorem 2.20 and Corollary 2.21 can be written in the following way.

**Corollary 2.22.** *Let $D$ be a CKI-digraph on at least 4 vertices and $H$ an independent set of vertices, then $D - H$ is not a tournament. Moreover, $D \ncong T_k - A(U)$ where $U$ is an induced subdigraph of a tournament $T_k$.*

**Theorem 2.23.** *Let $D$ be a CKI-digraph on at least 5 vertices. Then $D - \{x, y\}$ is not a tournament for every two vertices $x, y \in V(D)$.*

*Proof.* From Theorem 1.1 and Theorem 1.2, both $D$ and $Asym(D)$ are strongly connected. We assume by contradiction that there exist $x, y \in V(D)$ such that $D - \{x, y\}$ is a tournament which must be transitive. Therefore from Corollary 2.22, it follows that $\{x, y\}$ is not an independent set of $D$. Also $\{x,y\}$ is a cut set because $D-\{x,y\}$ is a transitive tournament. Let $(v_1,\dots,v_{n-2})$ be the hamiltonian path in $D-\{x,y\}$. Since $\{x,y\}$ is also a cut set in $Asym(D)$, it follows that $D$ has at least an asymmetric $v_{n-2}\{x,y\}$-arc and an asymmetric $\{x,y\}v_1$-arc. Without loss of generality suppose that $(v_{n-2},y)\in A(Asym(D))$. Then $(x,v_1)\in A(Asym(D))$, because otherwise $(y,v_1)\in A(Asym(D))$ implying that $D$ contains the induced triangle $(y,v_1,v_{n-2},y)$, which is a contradiction. Moreover, there must be a vertex $z\in V(D)$ such that $(y,z)\in A(Asym(D))$. If $z=v_i$, then the induced triangle $(y,v_i,v_{n-2},y)$, produces a contradiction, so $z=x$ and $(y,x)\in A(Asym(D))$.

Let $K_1$ be a kernel of $D-v_1$. Since $(v_1,v_i)\in A(Asym(D))$ for all $i\in\{2,\dots,n-2\}$, $v_i\notin K_1$ by Lemma 2.2. Thus, $K_1\subseteq\{x,y\}$ and since $(y,x)\in A(Asym(D))$, the kernel of $D-v_1$ is $K_1=\{x\}$. Then $(v_j,x)\in A(D)$ for all $j$ with $1<j\le n-2$. This implies that $(x,v_j)\in A(Sym(D))$ for all $j$ with $1<j\le n-2$ because otherwise $(x,v_1,v_j,x)$ is an induced $\vec{C}_3$ and $D$ is CKI, see Figure 4.

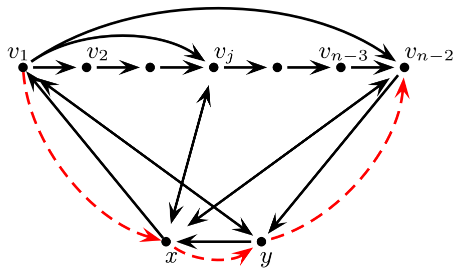


Figure 4: Case $K_1=\{x\}$.

Let $K_x$ be a kernel of $D-x$. By Lemma 2.2, $v_j\notin K_x$ for all $j$ with $1\le j\le n-2$, hence it is forced that $K_x=\{y\}$. Then $(v_j,y)\in A(D)$, for all $j$ with $1\le j\le n-2$ and by Lemma 2.2, $(y,x)\in A(Asym(D))$. Moreover, $(v_1,y)\in A(Sym(D))$, otherwise an induced triangle $\vec{C}_3$ is formed in $D$ by $(v_1,y,v,v_1)$, see Figure 4. Since $D$ has at least 5 vertices and $(v_{n-2},y)\in Asym(D)$, the set $\{v_1,v_{n-2},x,y\}$ induces a proper subdigraph of $D$ isomorphic to $\vec{C}_4(1,2)$ (the digraph of Figure 2) which is a contradiction because $\vec{C}_4(1,2)$ has no kernel.

Thus, $D-\{x,y\}$ is not a transitive tournament and the theorem is proved. $\square$

**Corollary 2.24.** *Let $D$ be a CKI-digraph on $n\ge 4$ vertices. Then the maximum tournament contained in $D$ has at most $n-3$ vertices.*

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