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Characterization of asymmetric CKI- and KP-digraphs with covering number at most 3[☆]

Hortensia Galeana-Sánchez[a], Mika Olsen[b,*]

[a] Instituto de Matemáticas, Universidad Nacional Autónoma de México, Circuito Exterior, Ciudad Universitaria, 04510 México DF, Mexico

[b] Departamento de Matemáticas Aplicadas y Sistemas, Universidad Autónoma Metropolitana - Cuajimalpa, Calle Artificios 40, col Hidalgo, 01120 México DF, Mexico

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ABSTRACT

A set $N \subseteq V(D)$ is said to be a kernel if N is an independent set and for every vertex $x \in (V(D) \setminus N)$ there is a vertex $y \in N$ such that $xy \in A(D)$. Let D be a digraph such that every proper induced subdigraph of D has a kernel. D is said to be *kernel perfect digraph* (KP-digraph) if the digraph D has a kernel and *critical kernel imperfect digraph* (CKI-digraph) if the digraph D does not have a kernel. In this paper we characterize the asymmetric CKI-digraphs with covering number at most 3. Moreover, we prove that the only asymmetric CKI-digraphs with covering number at most 3 are: $\overrightarrow{C}_3$, $\overrightarrow{C}_5$ and $\overrightarrow{C}_7(1,2)$. Several interesting consequences are obtained.

1. Introduction

For general concepts we refer the reader to [2,3]. The topic of domination in graphs has been widely studied by several authors, a very complete study of this topic is presented in [17,18]. The absorption in digraphs is the dual concept of domination, and it is defined as follows: Let D be a digraph, a set of vertices $S \subseteq V(D)$ is an absorbing set if for every vertex $w \in V(D) \setminus S$ there is an arc $wv \in A(D)$ with $v \in S$. Absorbing independent sets in digraphs (kernels in digraphs) have found many applications in different topics of mathematics (for instance [19,20,12,13,23]) and they have been studied by several authors, interesting surveys of kernels in digraphs can be found in [8,13].

Let D be a digraph such that every proper induced subdigraph of D has a kernel. D is said to be *kernel perfect digraph* (KP-digraph) if the digraph D has a kernel and *critical kernel imperfect digraph* (CKI-digraph) if the digraph D does not have a kernel.

The perfect graphs were introduced by the Strong Perfect Conjecture stated by C. Berge in 1960. A graph G is called a *perfect graph* if, for each induced subgraph H of G, the chromatic number of H is equal to the maximum number of pairwise adjacent vertices in H. This conjecture states that a graph G is perfect if and only if G contains neither C_{2n+1} nor the complement of C_{2n+1}, $n \geq 2$, as an induced subgraph and it was proved by M. Chudnovsky et al. (2006) [10]. Many authors have contributed to obtain nice properties and interesting characterizations of Perfect Graphs [4,22]. In 1986 C. Berge and P. Duchet conjectured that a graph G is perfect if and only if any orientation by sinks of G is a kernel perfect digraph. (If G is a graph, an orientation $\overrightarrow{G}$ of G is a digraph obtained from G by directing each edge of G in at least one of the two possible directions. An orientation $\overrightarrow{G}$ of G is called an *orientation by sinks* (or normal) if every semicomplete subgraph H of G has

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* Corresponding author.
E-mail addresses: olsen@correo.cua.uam.mx, olsen.mika@gmail.com (M. Olsen).

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an absorbing vertex in $\overrightarrow{G}[V(H)]$). This Conjecture was proved in [5,9] and it constructs an important bridge between two topics in graph theory: namely colorings and kernels.

Let D be a digraph, $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of D respectively. An arc $uv \in A(D)$ is called asymmetric if $vu \notin A(D)$. The *asymmetric part* of D, denoted by $Asym(D)$, is the subdigraph of D, with vertex set $V(D)$ and whose arcs are the asymmetric arcs of D. A *semicomplete digraph* is a digraph D such that there is at least one arc between any two vertices of $V(D)$.

The *covering number* of a digraph D, denoted $\sigma(D)$, is the minimum number of semicomplete digraphs of D that partition $V(D)$. Digraphs with a small covering number are a nice class of nearly tournament digraphs. The existence of kernels in the digraphs with covering number at most 3 has been studied by several authors, in particular by Berge [5], Maffray [21] and others [6,7,14,15].

In this paper, we study the CKI-digraphs D with covering number of D or $Asym(D)$ at most 3. For the case when the covering number of D or $Asym(D)$ is at most two, we use the connection between perfect graphs and it turns out, that the only CKI-digraphs with covering number at most two are orientations of perfect graphs. Hence, they are not orientations by sinks. In contrast, when the covering number is three, CKI-digraphs are not necessarily orientations of a perfect graph. Therefore, when the covering number of D or $Asym(D)$ is 3, we cannot use the connection between perfect graphs and the kernels. Also, we characterize the CKI-digraphs and the KP-digraphs that satisfy that the covering number of (the asymmetric part of) any strongly connected component is at most 2.

2. Definitions and preliminaries

Let D be a digraph, $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of D respectively. We denote the arc (u, v) by uv. For any $v \in V(D)$, we denote by $N^+(v)$ and $N^-(v)$ the *out*- and *in-neighborhood* of v in D respectively. All the paths, cycles and walks considered in this paper will be directed paths, cycles or walks of the digraph D. Let U, V be two disjoint subsets of $V(D)$, we denote by $(U, V) = \{uv \in A(D) : u \in U, v \in V\}$. If $U = \{u\}$ (resp. $V = \{v\}$), then (u, V) (resp. (U, v)) denotes the set of arcs (U, V).

A *tournament* T is a digraph such that there is exactly one arc between any two vertices of T. An *acyclic digraph* is a digraph without directed cycles. An acyclic tournament is called a *transitive tournament*. A vertex $v \in V(D)$ *absorbs* the vertex set $S \subset V(D)$ if $sv \in A(D)$ for every $s \in S$. A vertex $v \in V(D)$ is a *sink* of D if v absorbs the vertex-set $V(D) \setminus \{v\}$. A *sink ordering* of the vertex-set $V(D)$ is a sequence $(u_1, u_2, \dots, u_n)$, where $|V(D)| = n$, u_1 is a sink of D and u_i is a sink of $D \setminus \{u_1, u_2, \dots, u_{i-1}\}$ for every $1 < i < n$ (in case that such an ordering can be defined). A tournament with a sink ordering is a transitive tournament and in this case the sink ordering is unique, but this is not necessarily true for a semicomplete digraph with a sink ordering. Let U be a subset of $V(D)$. We denote by $D[U]$ the subdigraph of D induced by U. We say that a digraph D is H-free if D has no induced subdigraph isomorphic to H.

Let G be a graph. Following the notation of Berge and Duchet [5] an *orientation* $\vec{G}$ of G is the digraph obtained by changing each edge with an asymmetric arc or symmetric arc. Let D be a digraph. The *underlying graph* G_D of D is the graph obtained by changing each asymmetric arc by an edge and each pair of symmetric arcs by an edge. The underlying graph of a digraph is a simple graph. Let G be a graph, the graph $\overline{G}$ is the graph defined on the vertex-set $V(G)$ and $E(\overline{G}) = \{\{u, v\} : \{u, v\} \notin E(G)\}$.

We will need the following results.

Proposition 1 ([16]). *If D is not KP, then D has an induced CKI-subdigraph.*

Remark 1. If D is a CKI-digraph (or a KP-digraph), then D has no proper induced CKI-subdigraph. In particular $Asym(D)$ has no proper subdigraph isomorphic to $\overrightarrow{C}_3$.

Theorem 1 ([11,16]). *Let D be a CKI-digraph. Then $Asym(D)$ is strongly connected.*

Theorem 2 ([2]). *If the tournament T is strongly connected, then T is pancyclic.*

A graph G is called a *perfect graph* if, for each subgraph H of G the chromatic number of H is equal to the maximum number of pairwise adjacent vertices in H.

The following theorem is well known. We use Theorem 3 throughout this paper without mention it.

Theorem 3 ([3]). *A graph G is perfect if and only if $\overline{G}$ is a perfect graph.*

The following Theorem is a direct consequence of the results in [5,9].

Theorem 4 ([5,9]). *A graph G is perfect if and only if any orientation by sinks of G is a KP-digraph.*

The *covering number* of a digraph D, denoted $\sigma(D)$, is the minimum number of semicomplete subdigraphs of D that partition $V(D)$. Let D be a digraph with covering number σ. Then there is a partition of $V(D)$ into σ semicomplete subdigraphs of D, we call such a partition a covering set of D.

Remark 2. If D is a CKI-digraph with covering number σ, then the order of a kernel of $D - x$ is at most σ for every $x \in V(D)$.

Let $\mathbb{Z}_m$ be the cyclic group of integers modulo $m(m \geq 1)$ and J a nonempty subset of $\mathbb{Z}_m \setminus \{0\}$. A *circulant* (or *rotational*) *digraph* $\overrightarrow{C}_m(J)$ is defined by $V(\overrightarrow{C}_m(J)) = \mathbb{Z}_m$ and

$$A(\overrightarrow{C}_m(J)) = \{(i, j) : i, j \in \mathbb{Z}_m, j - i \in J\}.$$

Recall that the circulant digraphs are regular and they are vertex transitive.

3. CKI-digraphs with covering number at most 2

In this section we characterize the asymmetric CKI-digraphs with covering number at most 2 and the CKI-digraphs (resp. KP-digraphs) for which $Asym(D)$ has covering number at most 2. As a consequence, we characterize the KP-digraphs with the property that each strongly connected component W satisfies that $Asym(W)$ has covering number at most 2.

Let D be a digraph with covering number 2. A covering set of D induces a partition of $V(D)$ into two semicomplete digraphs. If $\sigma(Asym(D)) = 2$, then a covering set of $Asym(D)$ induces a partition of $V(D)$ into two tournaments and the set of symmetric arcs of D is a subset of the arcs of $[U, V]$. Therefore $Sym(D)$ is a bipartite digraph.

As a consequence of the Proposition 1, we have the following.

Lemma 1. *Let D be an asymmetric CKI-digraph with covering number at least two. If $U \subset V(D)$ such that $D[U]$ is a tournament, then $D[U]$ is a transitive tournament.*

By Lemma 1, a covering set of $Asym(D)$ induces a partition into transitive tournaments.

Theorem 5 (*[3]*)**.** *A semicomplete digraph D is kernel perfect if and only if each directed cycle has at least one symmetric arc.*

As a consequence of Theorem 5 and the fact that $\overrightarrow{C}_m\left(1, \pm 2, \pm 3, \ldots, \pm\lfloor \frac{m}{2} \rfloor\right)$ are CKI-digraphs [16], we have the following.

Theorem 6. *The only CKI-digraphs with covering number* 1 *are the circulant digraphs*

$$\overrightarrow{C}_m\left(1, \pm 2, \pm 3, \ldots, \pm\lfloor \frac{m}{2} \rfloor\right).$$

The following result is a consequence of Theorems 3 and 4.

Proposition 2. *Let D be a digraph with $\sigma(D) \leq 2$. Then D is a KP-digraph if and only if D is a $\overrightarrow{C}_3$-free digraph.*

Proof. If D is a KP-digraph, then D is $\overrightarrow{C}_3$-free, by Remark 1. Let D be a $\overrightarrow{C}_3$-free digraph with covering number two. In order to prove that D is a KP-digraph, we prove that D is oriented by sinks and that the underlying graph of D is a perfect graph. So, the result follows from Theorem 4.

Every semicomplete subdigraph of D has a sink, because D is $\overrightarrow{C}_3$-free, hence D is oriented by sinks. Let G_D be the underlying graph of D, clearly $\overline{G_D}$ is a bipartite graph or $\overline{K_p}$ (the complement of the complete graph on p vertices), and so $\overline{G_D}$ is perfect. So, G_D is a perfect graph and D is an orientation by sinks of G_D, and then, by Theorem 4, D is a KP-digraph. □

Corollary 1. *There are no CKI-digraphs with covering number* 2 *. Moreover, $\overrightarrow{C}_3$-free digraphs with covering number* 2 *are kernel perfect.*

Corollary 2. *Let D be a digraph with $\sigma(Asym(D)) \leq 2$. Then*

(i) *D is a CKI-digraph if and only if $D \cong \overrightarrow{C}_3$ or $D \cong \overrightarrow{C}_4(1, 2)$.*
(ii) *D is a KP-digraph if and only if D has no induced subdigraph isomorphic to $\overrightarrow{C}_3$ nor isomorphic to $\overrightarrow{C}_4(1, 2)$.*

Proof. If D is a digraph with $\sigma(Asym(D)) \leq 2$, then $\sigma(D) \leq 2$.

(i) By Theorem 6, $\overrightarrow{C}_3$ and $\overrightarrow{C}_4(1, 2)$ are CKI-digraphs. Let D be a CKI-digraph with $\sigma(Asym(D)) \leq 2$. By Corollary 1, $\sigma(D) =$ 1 and by Theorem 6, it follows that $D \cong \overrightarrow{C}_m\left(1, \pm 2, \pm 3, \ldots, \pm\lfloor \frac{m}{2} \rfloor\right)$, since $\sigma(Asym(D)) \leq 2$, $m < 5$ and we are done.

(ii) Since $\sigma(Asym(D)) \leq 2$, an induced subdigraph of D has covering number at most 2. If D has no induced subdigraph isomorphic to $\overrightarrow{C}_3$ nor isomorphic to $\overrightarrow{C}_4(1, 2)$, then by (i), D has no induced CKI-digraphs and D is a KP-digraph by Proposition 1. If D is a KP-digraph, then by Proposition 1, D has no induced subdigraph isomorphic to $\overrightarrow{C}_3$ nor isomorphic to $\overrightarrow{C}_4(1, 2)$. □

Corollary 3. *Let D be a digraph such that the covering number of the asymmetric part of every strongly connected component is at most* 2*. Then*

(i) *D is a CKI-digraph if and only if $D \cong \overrightarrow{C}_3$ or $D \cong \overrightarrow{C}_4(1, 2)$.*
(ii) *D is a KP-digraph if and only if D has no induced subdigraph isomorphic to $\overrightarrow{C}_3$ nor isomorphic to $\overrightarrow{C}_4(1, 2)$.*

Proof. Let D be a digraph such that the covering number of the asymmetric part of every strongly connected component is at most 2.

(i) The circulant digraphs $\overrightarrow{C}_3$ and $\overrightarrow{C}_4(1,2)$ are both CKI-digraphs by Theorem 6. Let D be a CKI-digraph. By Theorem 1, D is strongly connected, thus $\sigma(Asym(D)) \leq 2$ and by Corollary 2, we are done.

(ii) Let D be a digraph as in the hypothesis without induced subdigraphs isomorphic to $\overrightarrow{C}_3$ nor isomorphic to $\overrightarrow{C}_4(1,2)$. Suppose, for a contradiction, that D is not KP. By Theorem 1, D has an induced CKI-subdigraph H and H has covering number at least 2 because D is $\overrightarrow{C}_3$-free. By Theorem 1, $Asym(H)$ is strongly connected, so H is a subdigraph of a strongly connected component W of D. Thus, $\sigma(Asym(H)) = 2$ and $H \cong \overrightarrow{C}_3$ or $H \cong \overrightarrow{C}_4(1,2)$ by Corollary 2 which contradicts the hypothesis. □

As a summary for CKI-digraphs with covering number of asymmetric part at most two it was proved that there are exactly two CKI digraph with covering number at most two: $\overrightarrow{C}_3$ and $\overrightarrow{C}_4(1,2)$. These two digraphs shows that $\overrightarrow{C}_3$ is the only asymmetric digraph with covering number at most two.

4. Asymmetric CKI-digraphs with covering number 3

In this section we prove that the only two asymmetric CKI-digraphs with covering number 3 are $\overrightarrow{C}_5$ and $\overrightarrow{C}_7(1,2)$. It is easy to see that both $\overrightarrow{C}_5$ and $\overrightarrow{C}_7(1,2)$ are asymmetric digraphs with covering number 3 and that $\overrightarrow{C}_5$ is a CKI-digraph. It was proved by Duchet [11] that $\overrightarrow{C}_7(1,2)$ is a CKI-digraph.

Throughout this paper we use the following notations for asymmetric CKI-digraphs D with covering number three. Let U, V, W be a covering set of D since D is asymmetric, by Lemma 1, $D[U]$, $D[V]$ and $D[W]$ are transitive tournaments. Let $(u_n, u_{n-1}, \ldots, u_1)$, $(v_m, v_{m-1}, \ldots, v_1)$ and $(w_l, w_{l-1}, \ldots, w_1)$ be the sink orderings of U, V and W respectively (notice that u_1, v_1 and w_1 are the sinks of $D[U]$, $D[V]$ and $D[W]$ respectively).

In order to prove our main theorem, we analyze all the possibilities for $D[\{u_1, v_1, w_1\}]$. In Propositions 3 and 4 we analyze the possibilities for the case when $D[\{u_1, v_1, w_1\}]$ has at least two arcs and a sink or a source, in Proposition 5 we analyze the case when $D[\{u_1, v_1, w_1\}]$ is a path of length two and in Proposition 6, when $D[\{u_1, v_1, w_1\}]$ has exactly one arc.

The following remark is a consequence of Theorem 1. We use Remark 3 throughout this paper without mentioning it.

Remark 3. Let D be an asymmetric CKI-digraph with covering number three, and let U, V, W be a covering set of D into transitive tournaments, with u_1, v_1 and w_1 the sinks of U, V and W respectively. Then $d^+(u_1) \neq 0$, $d^+(v_1) \neq 0$ and $d^+(w_1) \neq 0$.

Since D is $\overrightarrow{C}_3$-free, we have the following.

Lemma 2 (*[1]*)**.** *Let D be a CKI-digraph and let K be a kernel of $D - \{v\}$ where v is a vertex of D. Then there is no arc from v to K and there is some arc from K to v.*

Lemma 3. *Let D be an asymmetric CKI-digraph with covering number three, and let U, V, W be a covering set of D. Let u_i, v_j be an independent set that absorbs the vertices of $U \setminus u_i \cup V \setminus v_j$ and suppose that $w_1u_i \in A(D)$. If α is the smallest integer such that $w_\alpha u_i \notin A(D)$, then $v_jw_\alpha \in A(D)$.*

Proof. Let D be a digraph that satisfies the conditions of the Lemma, and $u_i, v_j \in V(D)$ such that u_i, v_j absorbs the vertices of $U \setminus u_i \cup V \setminus v_j$. If $wu_i \in A(D)$ for every $w \in W$, then $\{u_i, v_j\}$ is a kernel of D, which is a contradiction. Let α be the smallest integer such that $w_\alpha u_i \notin A(D)$. Then $\{u_i, w_\alpha\}$ is an independent set, by the path (w_α, w_1, u_i). If $\{v_j, w_\alpha\}$ is an independent set, then $K = \{u_i, v_j, w_\alpha\}$ is a kernel of D, so $\{v_j, w_\alpha\}$ is not an independent set.

In order to prove that $v_jw_\alpha \in A(D)$, we suppose, for a contradiction, that $w_\alpha v_j \in A(D)$. Then $K_1 = \{u_i, v_j\}$ absorbs the vertices of $U \setminus u_i \cup V \setminus v_j \cup \{w_1, w_2, \ldots, w_\alpha\}$. Moreover, since D is $\overrightarrow{C}_3$-free, $w_1u_i \in A(D)$ and $w_\alpha v_j \in A(D)$, then

$$u_iw_k, v_jw_k \notin A(D) \quad \text{for } \alpha < k \leq l. \tag{1}$$

If there is a vertex $w \in W$ such that $\{u_i, v_j, w\}$ is an independent set, then let β be the smallest integer such that $\{u_i, v_j, w_\beta\}$ is an independent set. By the choice of α we have that $\beta > \alpha$, and by the choice of β and by (1), there is an (w_k, K_1)-arc for every $k, \alpha < k \leq \beta$, which lead us to the contradiction that K_1 is a kernel of D. Then for every $k, \alpha < k \leq n$, there is an arc between w_k and some vertex in K_1 and by (1), this arc must be an (w_k, K_1)-arc. Thus K_1 is a kernel of D, which is a contradiction, so $w_\alpha v_j \notin A(D)$, and since $\{v_j, w_\alpha\}$ is not an independent set, then $v_jw_\alpha \in A(D)$. □

Proposition 3. *Let D be an asymmetric CKI-digraph with covering number three. If $|A(D[\{u_1, v_1, w_1\}])| \geq 2$, then $D[\{u_1, v_1, w_1\}]$ has no sink.*

Proof. Let D be an asymmetric CKI-digraph with covering number three, and let U, V, W be a covering set of D in tournaments, with u_1, v_1 and w_1 the sinks of U, V and W respectively. Suppose, for a contradiction, that $D[\{u_1, v_1, w_1\}]$ has a sink. Without loss of generality we assume that $\{u_1v_1, w_1v_1\} \subseteq A(D)$. If $u_iv_1, w_jv_1 \in A(D)$ for $1 < i \leq l$ and $1 < j \leq n$, then $K = \{v_1\}$ is kernel of D which contradicts that D is a CKI-digraph. By symmetry, we may assume that there is a vertex $u \in U$ such that $uv_1 \notin A(D)$, let α be the smallest integer such that $u_\alpha v_1 \notin A(D)$. Then, $\{u_\alpha, v_1\}$ is an independent set, by the path (u_α, u_1, v_1). Let $K_1 = \{u_\alpha, v_1\}$; K_1 absorbs the vertices of $U \setminus u_\alpha \cup V \setminus v_1$. If $w_j v_1 \in A(D)$ for $1 < j \leq n$, then K_1 is kernel of D. Otherwise, let β be the smallest integer such that $w_\beta v_1 \notin A(D)$. Then $\{v_1, w_\beta\}$ is an independent set, by the path (w_β, w_1, v_1). By Lemma 3, $u_\alpha w_\beta \in A(D)$. Analogously if we consider $K_2 = \{w_\beta, v_1\}$, then K_2 absorbs the vertices of $V \setminus v_1 \cup W \setminus w_\beta$ and by Lemma 3, $w_\beta u_\alpha \in A(D)$, which contradicts that D is asymmetric. So, $D[\{u_1, v_1, w_1\}]$ has no sink. □

Proposition 4. *Let D be an asymmetric CKI-digraph with covering number three. If $|A(D[\{u_1, v_1, w_1\}])| \geq 2$, then $D[\{u_1, v_1, w_1\}]$ has no source.*

Proof. Let D be an asymmetric CKI-digraph with covering number three, and let U, V, W be a covering set of D in tournaments, with u_1, v_1 and w_1 the sinks of U, V and W respectively.

Suppose, for a contradiction, that $D[\{u_1, v_1, w_1\}]$ has a source. Without loss of generality we assume that $\{v_1u_1, v_1w_1\} \subseteq A(D)$, if $\{u_1, w_1\}$ is not independent, then by Proposition 3 we are done. Therefore $A(D[u_1, v_1, w_1]) = \{v_1u_1, v_1w_1\}$. Since D is asymmetric and $\overrightarrow{C}_3$-free, then

$$u_1v_i,\ w_1v_i \notin A(D) \quad \text{for } 1 \leq i \leq m. \tag{2}$$

Let $K_1 = \{u_1, w_1\}$. K_1 absorbs the vertices of $U \setminus u_1 \cup W \setminus w_1 \cup \{v_1\}$. If there is a vertex $v \in V$ such that $\{u_1, v, w_1\}$ is an independent set, then let α be the smallest integer such that $\{u_1, v_\alpha, w_1\}$ is an independent set. In this case K_1 absorbs the vertex set $(U \setminus u_1) \cup (W \setminus w_1) \cup \{v_1, v_2, \ldots, v_{\alpha-1}\}$, and $\{u_1, v_\alpha, w_1\}$ is a kernel of D, which is a contradiction. So for every vertex $v_i \in V$ there is an arc between v_i and some vertex in K_1. By (2) it must be an (v_i, K_1)-arc and K_1 is a kernel of D, which is a contradiction. So, $D[\{u_1, v_1, w_1\}]$ has no source. □

Lemma 4. *Let D be an asymmetric CKI-digraph with covering number three, with $|U| = n$, $|V| = m$ and $|W| = l$. If $D[\{u_1, v_1, w_1\}]$ is a path of length 2 , then there exists a covering set of D in tournaments U', V', W' with $|U'|, |V'|, |W'| \geq 2$.*

Proof. By Remark 3, the CKI-digraph D satisfies that $d^+(w_1) > 0$. If $n = 1$, then $d^+(w_1) = 0$ because $(w_1, \{u_1\} \cup V \cup W) = \emptyset$, so $n > 1$. If $m = 1$, then $\{u_1, w_1\}$ is kernel of D, so $m > 1$.

Suppose for a contradiction that $|W| = 1$. We will construct a covering set with the required properties. Let N_w be the kernel of $D - \{w_1\}$. Since $(v_1, U \cup V) = \emptyset$, then $v_1 \in N_w$. Let α be the minimum integer such that $w_1u_\alpha \in A(D)$ (such an α does exist because $d^+(w_1) > 0$). By the 4-cycle $(u_1, v_1, w_1, u_\alpha, u_1)$, it follows that $\{u_\alpha, v_1\}$ is independent. Moreover, $u_iv_1 \in A(D)$ for $i < \alpha$, so $N_w = \{u_\alpha, v_1\}$, which is a contradiction because in this case N_w is a kernel of D. □

Proposition 5. *Let D be an asymmetric CKI-digraph with covering number three. If $D[\{u_1, v_1, w_1\}]$ is a path of length 2 , then $D \cong \overrightarrow{C}_5$ or $D \cong \overrightarrow{C}_7(1, 2)$.*

Proof. Let D be an asymmetric CKI-digraph with covering number three, and let U, V, W be a covering set of D in tournaments, with u_1, v_1 and w_1 the sinks of U, V and W respectively and without loss of generality we assume that $A(D[\{u_1, v_1, w_1\}]) = \{u_1v_1, v_1w_1\}$ (notice that the vertex set $\{u_1, w_1\}$ is an independent set). By Lemma 4, we may assume that $|U| = n$, $|V| = m$, $|W| = l$ and $n, m, l \geq 2$. By the paths (u_i, u_1, v_1) and (v_i, v_1, w_1),

$$\text{(a)}\ v_1u_i \notin A(D) \quad \text{for every } 1 \leq i < n, \qquad \text{(b)}\ \ w_1v_i \notin A(D) \quad \text{for every } 1 \leq i < m. \tag{3}$$

Since D is a CKI-digraph, the digraph $D - \{w_l\}$ does have a kernel. Let N_w be a kernel of $D - \{w_l\}$. By Lemma 2, $w_i \notin N_w$. By (3)(a) and the fact that v_1 is sink of $D[V]$, $v_1 \in N_w$.

Since $|W| > 1$, $w_1 \notin N_w$ and then by the arc v_1w_1, there is a vertex $u_\alpha \in N_w \cap U$ such that $w_1u_\alpha \in A(D)$ and $\alpha > 1$. Then $N_w = \{u_\alpha, v_1\}$ and by the definition of N_w, Lemma 1 and the arc v_1w_1, we have that $u_iv_1 \in A(D)$ for every $i < \alpha$ and $w_1u_\alpha \in A(D)$. Since N_w is a kernel of $D - \{w_l\}$, the path (w_l, w_1, u_α) and Lemma 2, it follows that

$$\text{(a)}\ \{u_\alpha, w_l\} \text{ is independent}, \qquad \text{(b)}\ \ u_iv_1 \in A(D) \quad \text{for } 1 \leq i < \alpha, \qquad (c)\ \ v_1w_l \in A(D). \tag{4}$$

By the path (v_1, w_l, w_i) and the definition of N_w, it follows that

$$w_iu_\alpha \in A(D) \quad \text{for } i < l. \tag{5}$$

Consider the digraph $D - \{v_1\}$. Since D is a CKI-digraph, the digraph $D - \{v_1\}$ does have a kernel. Let N_v be a kernel of $D - \{v_1\}$. By assumption, (4)(c) and the definition of N_v,

$$w_1, w_l \notin N_v. \tag{6}$$

By (3)(b), w_1 must be absorbed by some $u_i \in U$, $i > 1$. Let $u_\beta = N_v \cap U$ and $w_1u_\beta \in A(D)$. By (4)(c), for $i < \alpha$, the path (u_i, v_1, w_1), leads to $\beta \geq \alpha > 1$. Note that $N_v \neq \{u_\beta\}$, because $\beta > 1$ and $u_1u_\beta \notin A(D)$.

By the path (w_i, u_α, u_1), $u_1w_i \notin A(D)$ for every $i < l$ and so, by (6), $N_v \cap V \neq \emptyset$, else $(u_1, N_v) = \emptyset$. Let $N_v \cap V = \{v_\gamma\}$. By the choice of N_v, $\gamma > 1$. By the definition of N_v and the path (v_γ, v_1, w_l),

$$\text{(a)}\ u_1v_\gamma \in A(D), \qquad \text{(b)}\ w_lv_\gamma \notin A(D). \tag{7}$$

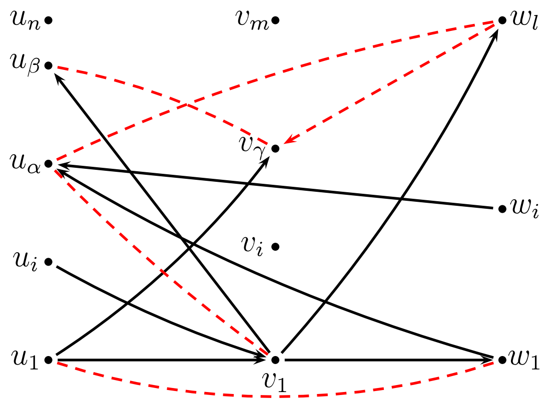

Fig. 1. Proposition 5.

If $\alpha = \beta$, then w_l is not absorbed by the vertex set $\{u_\beta, v_\gamma\}$. By (6), w_l must be absorbed by N_v, so $N_v \cap W \neq \emptyset$. Let $N_v \cap W = \{w_\delta\}$, notice that $1 < \delta < l$, by (6). In this case, $N_v = \{u_\beta, v_\gamma, w_\delta\}$ which contradicts (5).

We may assume that $\alpha < \beta$.

By the 4-cycle $(u_i, v_1, w_1, u_\beta, u_i)$, for $i < \alpha$, the set $\{u_\beta, v_1\}$ is independent and

$$\{u_i, w_1\} \text{ is independent, for } i < \alpha. \tag{8}$$

In Fig. 1, we show the arcs that must be in the digraph D. With dashed lines we indicate the independent sets as well as the arcs that are not arcs in D.

Claim 1. *If $N_v = \{u_\beta, v_\gamma\}$, then $D \cong \overrightarrow{C}_5$ or $D \cong \overrightarrow{C}_7(1, 2)$.*

Let $N_v = \{u_\beta, v_\gamma\}$. By (7)(b), $w_l u_\beta \in A(D)$, $u_i v_\gamma \in A(D)$, for every $i < \beta$; and from the 4-cycle $(u_\alpha, v_\gamma, v_1, w_1, u_\alpha)$ it follows that $\{v_\gamma, w_1\}$ is independent. If $v_\gamma w_l \notin A(D)$, then $\{v_\gamma, w_l\}$ is independent by the path (v_γ, v_1, w_l). In this case $(u_\beta, u_\alpha, v_\gamma, v_1, w_l, u_\beta)$ is an induced 5-cycle and by Remark 1, $D \cong \overrightarrow{C}_5$.

So, we may assume that $v_\gamma w_l \in A(D)$. By the 4-cycle $(w_l, u_\beta, u_1, v_1, w_l)$ it follows that $\{u_1, w_l\}$ is independent. In this case $(u_\beta, u_\alpha, u_1, v_\gamma, v_1, w_l, w_1, u_\beta)$ induces a $\overrightarrow{C}_7(1, 2)$ and by Remark 1, $D \cong \overrightarrow{C}_7(1, 2)$. And Claim 1 is true. □

We may assume that $N_v \cap W \neq \emptyset$. Let $N_v = \{u_\beta, v_\gamma, w_\delta\}$. By (6), $1 < \delta < l$ and hence $w_\delta u_\alpha \in A(D)$. By the path (v_1, w_l, w_δ) and the definition of N_v, $\{v_1, w_\delta\}$ is independent. Hence, by definition of N_v and the arc $w_\delta u_\alpha$,

$$\text{(a) } u_1 v_\gamma \in A(D), \qquad \text{(b) } u_\alpha v_\gamma \in A(D). \tag{9}$$

By the 4-cycle $(u_\alpha, v_\gamma, v_1, w_1, u_\alpha)$, $\{v_\gamma, w_1\}$ is independent.

If $v_\gamma w_l \notin A(D)$, then $\{v_\gamma, w_l\}$ is independent by the path (v_γ, v_1, w_l). In this case, $(u_\alpha, v_\gamma, v_1, w_l, w_\delta, u_\alpha)$ is an induced 5-cycle and by Remark 1, $D \cong \overrightarrow{C}_5$. We may assume that $v_\gamma w_l \in A(D)$.

If $w_l u_\beta \in A(D)$, then $(u_1, v_1, w_l, u_\beta, u_1)$ is a 4-cycle and $\{u_1, w_l\}$ is independent. In this case, by (9)(a), $(u_\beta, u_\alpha, u_1, v_\gamma, v_1, w_l, w_1, u_\beta)$ induces a $\overrightarrow{C}_7(1, 2)$ and by Remark 1, $D \cong \overrightarrow{C}_7(1, 2)$. We may assume that $w_l u_\beta \notin A(D)$. By the path (w_l, w_1, u_β), $\{u_\beta, w_l\}$ is independent.

If $u_1 w_l \notin A(D)$, then $\{u_1, w_l\}$ is independent by the path (u_1, v_1, w_l). In this case, by (9)(a), $(u_\beta, u_1, v_\gamma, w_l, w_1, u_\beta)$ is an induced 5-cycle and by Remark 1, $D \cong \overrightarrow{C}_5$. We may assume that

$$u_1 w_l \in A(D). \tag{10}$$

By the 4-cycle $(u_1, w_l, w_i, u_\alpha, u_1)$, $\{u_1, w_i\}$ is independent for $i < l$.

Consider the digraph $D - \{u_1\}$. Since D is a CKI-digraph, the digraph $D - \{u_1\}$ does have a kernel. Let N_u be a kernel of $D - \{u_1\}$.

By assumption, (9)(b) and (10),

$$v_1, v_\gamma, w_l \notin N_u. \tag{11}$$

By (3)(a) and the fact that v_1 is sink of V it follows that $(v_1, \{U \cup V\}) = \emptyset$, in this case, $N_u \cap W \neq \emptyset$ by (11). Let $w_\epsilon \in N_u$ for some $\epsilon < l$ and $v_1 w_\epsilon \in A(D)$. By (7)(a) and the paths $(w_\epsilon, u_\alpha, v_\gamma)$ and (u_i, u_1, v_γ), it follows that $(v_\gamma, \{U \cup \{w_\epsilon\}\}) = \emptyset$. Then $N_u \cap V \neq \emptyset$, by (11) and $v_\zeta \in N_u$ for some $\zeta < \gamma$.

By (5), $w_\epsilon u_\alpha \in A(D)$, so $u_\alpha \notin N_u$ and then, (u_α, N_u) must be non empty. If $u_\alpha v_\zeta \in A(D)$, then by the paths $(w_\delta, u_\alpha, v_\zeta)$ and $(u_\beta, u_\alpha, v_\zeta)$, it follows that $(v_\zeta, \{u_\beta, w_\delta\}) = \emptyset$, which contradicts that N_v is a kernel of $D - \{v_1\}$, because $\zeta > 1$ by (11). So $u_\alpha v_\zeta \notin A(D)$. By (4)(a) and (5), it follows that $(u_\alpha, \{v_\zeta, w_\epsilon\}) = \emptyset$. So, $N_u \cap U \neq \emptyset$ and let $u_\eta \in N_u$ for some $\eta < \alpha$. Hence $N_u = \{u_\eta, v_\zeta, w_\epsilon\}$.

If $\epsilon > 1$, then by (3)(b) and (8)(b), it follows that $(w_1, N_u) = \emptyset$, which is a contradiction, so $\epsilon = 1$ and $N_u = \{u_\eta, v_\zeta, w_1\}$.

We will prove that $N = \{u_1, v_\zeta, w_1\}$ is kernel of D. By the definition of the kernel N_u of $D - \{u_1\}$ and the path (u_1, v_γ, v_ζ) it follows that $N = \{u_1, v_\zeta, w_1\}$ is independent. Moreover, N absorbs $U \cup \{v_i : i > \zeta\} \cup W$. In order to prove that N absorbs the vertices v_i, $1 \le i < \zeta$, we prove that $v_i w_1 \in A(D)$ for every $1 \le i < \zeta$. By definition of N_v (kernel of $D - \{v_1\}$) and the fact that $w_\delta u_\alpha \in A(D)$, it follows that $u_i v_\gamma \in A(D)$ for every $i < \alpha$ and so, by the path (u_η, v_γ, v_i) for $i < \zeta$, we have that $v_i u_\eta \notin A(D)$. By the definition of N_u, it follows that $v_i w_1 \in A(D)$ for every $1 \le i < \zeta$. Hence $N = \{u_1, v_\zeta, w_1\}$ is a kernel of D, which contradicts that D is a CKI-digraph.

So, we are done. □

Lemma 5. *Let D be an asymmetric CKI-digraph with covering number three, with $|U| = n$, $|V| = m$ and $|W| = l$. If $|A(D[\{u_1, v_1, w_1\}])| = 1$, then $D \cong \overrightarrow{C}_5$ or $D \cong \overrightarrow{C}_7(1, 2)$ or there exists a covering set of D in tournaments U', V', W' with $|U'|, |V'|, |W'| \ge 2$.*

Proof. Without loss of generality, we assume that $A(D[\{u_1, v_1, w_1\}]) = \{u_1 v_1\}$. If $v_1 u_i \in A(D)$, then (v_1, u_i, u_1, v_1) is a C_3, moreover $N^+(v_1) \subset W$ because v_1 is a sink of V. Since $v_1 w_1 \notin A(D)$ and $d^+(v_1) > 0$, then $l > 1$.

If $n = 1$, then $\{v_1, w_1\}$ is a kernel of D. So $n > 1$.

Suppose by contradiction that $m = 1$.

Claim 2. *If $n = 2$, then $D \cong \overrightarrow{C}_5$.*

The set $\{u_2, v_1\}$ is independent, else the covering number of D is two. Since $\{u_1, w_1\}$ and $\{v_1, w_1\}$ are independent sets and $d^+(w_1) > 0$, then $w_1 u_2 \in A(D)$.

Let N_1 be the kernel of $D - \{u_1\}$. In this case $v_1 \notin N_1$, so, v_1 must be absorbed by N_1, and then $N_1 \cap W = \emptyset$. Let $w_\alpha \in N_1$. Since $\{v_1, w_1\}$ is independent, $\alpha > 1$. In this case w_1 must be absorbed by N_1 and $N_1 \cap U \ne \emptyset$. Since $\{u_1, w_1\}$ is independent, $u_2 \in N_1$ and $N_1 = \{u_2, w_\alpha\}$. By the definition of N_1, $\{u_2, w_\alpha\}$ is an independent set. By Lemma 2 and the path (u_1, v_1, w_α), $\{u_1, w_\alpha\}$ is an independent set. In this case $(u_2, u_1, v_1, w_\alpha, w_1, u_2)$ is a induced 5-cycle and by Remark 1, $D \cong \overrightarrow{C}_5$. □

Thus, we assume that $n > 2$.

If $u_i v_1 \in A(D)$ for every $i \le n$, then the covering number of D is two, which is a contradiction. Let β be the smallest integer such that $u_\beta v_1 \notin A(D)$, then $\{u_\beta, v_1\}$ is independent. If $\beta < n$, then $U' = \{w_1, w_2, \ldots, w_l\}$, $V' = \{u_\beta, \ldots, u_n\}$, $W' = \{v_1, u_1, u_2, \ldots, w_{\beta-1}\}$ is a covering set with $|U'| > 1$, $|V'| > 1$ and $|W'| > 1$ and $A(D[u'_1, v'_1, w'_1]) = (u'_1, v'_1)$.

Thus we may assume that $\beta = n$.

Let N_w be kernel of $D - \{w_l\}$, then $N_w \cap W = \emptyset$. In this case $v_1 \in N_w$. Since $w_1 v_1 \notin A(D)$, $N_w \cap U \ne \emptyset$ and $N_w = \{u_n, v_1\}$ because $u_i v_1 \in A(D)$ for every $i < n$. Furthermore, $w_1 u_n \in A(D)$. By Lemma 2 and the path (w_l, w_1, u_n), $\{u_n, w_l\}$ is an independent set and $v_1 w_l \in A(D)$. By the path (v_1, w_l, w_i), $w_i v_1 \notin A(D)$ for $i < l$. By the definition of N_w, it follows that $w_i u_n \in A(D)$ for $i < l$. If $u_1 w_l \notin A(D)$, then $(u_n, u_1, v_1, w_l, w_1, u_n)$ is an induced $\overrightarrow{C}_5$ and by Remark 1, $D \cong \overrightarrow{C}_5$. So $u_1 w_l \in A(D)$ and $U' = \{v_1, u_2, \ldots, u_{n-1}\}$, $V' = \{w_l, u_1\}$, $W' = \{u_n, w_1, w_2, \ldots, w_{l-1}\}$ is a covering set with the property that $|U'|, |V'|, |W'| \ge 2$ and $A(D[u'_1, v'_1, w'_1]) = (u'_1, v'_1)$. □

Proposition 6. *Let D be an asymmetric CKI-digraph with covering number three. If $|A(D[\{u_1, v_1, w_1\}])| = 1$, then $D \cong \overrightarrow{C}_5$ or $D \cong \overrightarrow{C}_7(1, 2)$.*

Proof. Let D be an asymmetric CKI-digraph with covering number three, and let U, V, W be a covering set of D in tournaments, with u_1, v_1 and w_1 the sinks of U, V and W respectively, in view of Lemma 5, we will assume that $|U|, |V|, |W| > 1$. Let $A(D[\{u_1, v_1, w_1\}]) = \{u_1 v_1\}$. Do note that $\{u_1, w_1\}$ and $\{v_1, w_1\}$ are independent sets. By the path (u_k, u_1, v_1),

$$(v_1, U) = \emptyset. \tag{12}$$

Let N_w be a kernel of $D - \{w_l\}$.

Claim 3. *$N_w = \{u_\alpha, v_1\}$ for some $\alpha > 1$.*

By Lemma 2, $w_i \notin N_w$, thus by Remark 2, $|N_w| < 3$. If $|N_w| = 1$, then $N_w = \{v_1\}$, by (12) and the fact that v_1 is sink of V. In this case $(w_1, N_w) = \emptyset$ and N_w is not a kernel of $D - \{w_l\}$ (by Lemma 5, $l > 1$), which is a contradiction, so $|N_w| = 2$.

By Lemma 2 and (12), it follows that $N_w = \{u_\alpha, v_1\}$ for some $\alpha > 1$, which proves Claim 3. □

By definition of N_w, $\{u_\alpha, v_1\}$ is independent. So,

$$\text{(a)}\ u_i v_1 \in A(D) \quad \text{for any } 1 \le i < \alpha, \qquad \text{(b)}\ w_1 u_\alpha \in A(D). \tag{13}$$

By Lemmas 2 and 5 and the path (w_l, w_1, u_α),

$$\text{(a)}\ \{u_\alpha, w_l\} \text{ is independent}, \qquad \text{(b)}\ v_1 w_l \in A(D). \tag{14}$$

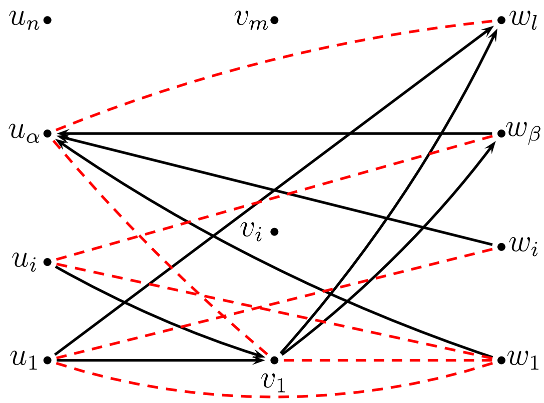

Fig. 2. Proposition 6.

By the path (v_1, w_l, w_i), it follows that $w_i v_1 \notin A(D)$. Then, by Lemma 5 and Claim 3,

$$w_i u_\alpha \in A(D) \quad \text{for any } 1 \le i < l. \tag{15}$$

If $u_1 w_l \notin A(D)$, then by the path (u_1, v_1, w_l), $\{w_l, u_1\}$ is independent. In this case $(u_1, v_1, w_l, w_1, u_\alpha, u_1)$ is an induced $\overrightarrow{C}_5$ and by Remark 1, $D \cong \overrightarrow{C}_5$.

Thus, we assume that

$$u_1 w_l \in A(D). \tag{16}$$

By the 4-cycle $(u_1, w_l, w_i, u_\alpha, u_1)$ and the paths (u_i, u_1, w_l) and (v_i, v_1, w_l),

$$\text{(a) } \{u_1, w_i\} \text{ is independent for } \ i < l \qquad \text{(b) } (w_l, U \cup V) = \emptyset. \tag{17}$$

Let N_1 be a kernel of $D - \{u_1\}$. By definition of N_1,

$$v_1, w_l \notin N_1. \tag{18}$$

By (12), (17)(b) and (18) there exists an integer β such that,

$$\text{(a) } w_\beta \in N_1 \quad \text{for some } 1 < \beta < l, \qquad \text{(b) } v_1 w_\beta \in A(D). \tag{19}$$

By (15)(b) and the definition of N_1,

$$\text{(a) } w_\beta u_\alpha \in A(D), \qquad \text{(b) } u_\alpha \notin N_1. \tag{20}$$

By the 4-cycle $(u_i, v_1, w_\beta, u_\alpha, u_i)$,

$$\{u_i, w_\beta\} \text{ is independent for } \quad 1 \le i < \alpha. \tag{21}$$

Let $i < \alpha$. If $w_1 u_i \in A(D)$, then $\{u_i, w_l\}$ is independent by the 4-cycle $(w_l, w_1, u_i, v_1, w_l)$. In this case $(u_\alpha, u_i, u_1, v_1, w_l, w_\beta, w_1, u_\alpha)$ induces a $\overrightarrow{C}_7(1, 2)$ and by Remark 1, $D \cong \overrightarrow{C}_7(1, 2)$.

Hence, we may assume that for every $i < \alpha$, $w_1 u_i \notin A(D)$, and by the path (w_1, u_α, u_i),

$$\{u_i, w_1\} \text{ is independent for } \quad i < \alpha. \tag{22}$$

In Fig. 2, we show the arcs that must be in the digraph D. With dashed lines we indicate the independent sets as well as the arcs that are not in D.

We will analyze separately the two cases: $\alpha < n$ or $\alpha = n$.

Case I $\alpha < n$.

Let N_u be a kernel of $D - \{u_n\}$. Then $U \cap N_u = \emptyset$. By (14)(a) and (15), $(u_\alpha, W) = \emptyset$, so there is a vertex $v_\gamma \in V \cap N_u$ such that $u_\alpha v_\gamma \in A(D)$. By definition of $N_w = \{u_\alpha, v_1\}$, $\gamma > 1$.

By the path $(u_n, u_\alpha, v_\gamma)$ and the definition of N_u, by the paths (v_γ, v_1, w_l) and (v_γ, v_1, w_β),

$$\text{(a) } \{u_n, v_\gamma\} \text{ is independent}, \qquad \text{(b) } w_l v_\gamma, w_\beta v_\gamma \notin A(D). \tag{23}$$

Since $\gamma > 1$ and $v_1 v_\gamma \notin A(D)$, by (12), there is a vertex $w_\delta \in W \cap N_u$ such that $v_1 w_\delta \in A(D)$. So $\delta > 1$, and $N_u = \{v_\gamma, w_\delta\}$. By the definition of N_u, $\{v_\gamma, w_\delta\}$ is independent. Moreover, $v_i w_\delta \in A(D)$ for $1 \le i < \gamma$ and

$$w_i v_\gamma \in A(D) \quad \text{for } 1 \le i < \delta. \tag{24}$$

By Lemma 2, (23)(a) and (17)(b),

$$\text{(a) } w_\delta u_n \in A(D), \qquad \text{(b) } 1 < \delta < l.$$

By definition of N_u and the path $(u_i, u_\alpha, v_\gamma)$, $v_\gamma u_i \notin A(D)$ for $i > \alpha$, and so, by the path (w_δ, u_n, u_i), with $i < n$, we have that $(U, w_\delta) = \emptyset$. Since N_u is kernel of $D - \{u_n\}$,

$$\text{(a) } u_i v_\gamma \in A(D), \quad \text{for } 1 \le i < n, \qquad \text{(b) } v_i u_1 \notin A(D) \quad \text{for } i < \gamma. \tag{25}$$

By (23)(b) and the 4-cycle $(v_1, w_\beta, u_\alpha, v_\gamma, v_1)$, $\{v_\gamma, w_\beta\}$ is independent and by (24), $\delta \le \beta$. By (25)(a), we have that $v_\gamma \notin N_1$ and so, by (24) and the definition of N_u, $(v_\gamma, U \cup W) = \emptyset$.

So, there is a vertex $v_\varepsilon \in V \cap N_1$ such that $1 < \varepsilon < \gamma$ by (19)(a) and (19)(b). By (25)(b), $\{u_1, v_\varepsilon\}$ is independent. By the definition of N_1 and Lemma 2, $N_1 \cap U \neq \emptyset$. Otherwise, by the fact that $\{u_1, v_\varepsilon\}$ is independent and (17)(a), it follows that $N' = \{u_1, v_\varepsilon, w_\beta\}$ independent. Moreover, N' is a kernel of D, which is a contradiction.

Let $N_1 = \{u_\chi, v_\varepsilon, w_\beta\}$. By the 4-cycles $(u_1, w_l, w_\delta, u_n, u_1)$, $(u_\alpha, v_\gamma, v_\varepsilon, w_\delta, u_\alpha)$ and $(u_i, v_1, w_\delta, u_\alpha, u_i)$, it follows that $\{u_n, w_l\}$, $\{u_\alpha, v_\varepsilon\}$ and $\{u_i, w_\delta\}$ are independent sets for $i < \alpha$. Then $\{u_n, v_1\}$ is independent by the 4-cycle $(u_1, v_1, w_\delta, u_n, u_1)$.

By (20)(a), (20)(b) and the independent set $\{u_\alpha, v_\varepsilon\}$, it follows that $\chi < \alpha$. By (22) and the path $(v_\varepsilon, w_\delta, w_1)$, $(w_1, N_1) = \emptyset$ which contradicts that N_1 is a kernel of $D - \{u_1\}$.

So, the case $\alpha < n$ leads to a contradiction.

Case 2 $\alpha = n$.

In this case, by Claim 3 and (14)(a), we obtain that $\{u_n, v_1\}$ and $\{u_n, w_l\}$ are independent sets, $N_w = \{u_n, v_1\}$, by (20) $u_n \notin N_1$ and by (15)(b),

$$w_i u_n \in A(D) \quad \text{for every } 1 \le i < l. \tag{26}$$

Since $(w_1, U \backslash \{u_n\} \cup \{w_\beta\}) = \emptyset$, there is a vertex $v_\gamma \in N_1 w_1 v_\gamma \in A(D)$. By (19)(b), $1 < \gamma$. By the 4-cycles $(w_1, v_\gamma, v_1, w_\beta, w_1)$ and $(w_1, v_\gamma, v_1, w_l, w_1)$ the following sets are independent,

$$\text{(a) } \{v_\gamma, w_\beta\}, \qquad \text{(b) } \{v_\gamma, w_l\}. \tag{27}$$

Claim 4. *If $u_n v_\gamma, v_\gamma u_1 \in A(D)$, then $D \cong \overrightarrow{C}_7(1, 2)$.*

Let $u_n v_\gamma \in A(D)$, $v_\gamma u_1 \in A(D)$, then $(u_n, v_\gamma, u_1, v_1, w_l, w_\beta, w_1, u_n)$ induces a $\overrightarrow{C}_7(1, 2)$ and by Remark 1, $D \cong \overrightarrow{C}_7(1, 2)$. □

If $N_1 = \{v_\gamma, w_\beta\}$, then $u_n v_\gamma \in A(D)$ by the definition of N_1 and by Lemma 2, $v_\gamma u_1 \in A(D)$ and $D \cong \overrightarrow{C}_7(1, 2)$ by Claim 4. So,

$$N_1 \cap (U \setminus \{u_1\}) \neq \emptyset. \tag{28}$$

Let N_v be a kernel of $D - \{v_m\}$. Then $N_v \cap V = \emptyset$.

Since $m > 1$ and $(v_1, U) = \emptyset$, then $w_\rho \in N_v$ for some $\rho > 1$ and $v_1 w_\rho \in A(D)$. Then $w_1 \notin N_v$ and since $(w_1, U \backslash \{u_n\}) = \emptyset$, then $u_n \in N_v$. Hence, by (15), $\rho = l$ and $N_v = \{u_n, w_l\}$. By Lemma 2 and the path (v_m, v_1, w_l) it follows that $\{v_m, w_l\}$ is independent and $u_n v_m \in A(D)$. By the path (u_n, v_m, v_i) and the definition of N_v, it follows that $v_i w_l \in A(D)$ for all $i < m$.

If $\gamma < m$, then by the path (u_n, v_m, v_γ) and by (27)(b) it follows that $(v_\gamma, N_v) = \emptyset$, which contradicts that N_v is a kernel of $D - \{v_m\}$. Then $\gamma = m$.

By definition of N_v, $u_i w_l \in A(D)$, for $i < n$, then by and Lemma 2, $\{v_m, w_l\}$ is independent and,

$$u_n v_m \in A(D). \tag{29}$$

By Claim 4 and the definition of N_1, we may assume that $\{u_1, v_m\}$ is independent. By the path (u_n, v_m, v_i), it follows that $v_i u_n \notin A(D)$. By the definition of N_v, it follows that $v_i w_l \in A(D)$. By the 4-cycle $(u_i, w_l, w_j, u_n, u_i)$,

$$\{u_i, w_j\} \text{ is independent for } \quad i < n \text{ and } j < l. \tag{30}$$

By (28) and (29),

$$N_1 = \{u_\chi, v_m, w_\beta\} \quad \text{for some } 1 < \chi < n. \tag{31}$$

Hence, there exists $u_2 \neq u_n$. Let N_2 be a kernel of $D - \{u_2\}$. Then $u_1, v_1, w_l \notin N_2$.

In this case, $v_y \in N_2$ for some $1 < y < m$, else $(u_1, N_2) = \emptyset$. Notice that, $v_m \notin N_2$ and then, $w_z \in N_2$ for some $1 < z < l$, else $(v_1, N_2) = \emptyset$. Also, $w_1 \notin N_2$ and then, $u_n \in N_2$, else $(w_1, N_2) = \emptyset$. Hence, $N_2 = \{u_n, v_y, w_z\}$. By (26), for $i < l$, $w_i u_n \in A(D)$ and since $w_l \notin N_2$, then $N_2 \cap W = \emptyset$, which contradicts that $N_2 = \{u_n, v_y, w_z\}$.

So Case 2, is settled. □

As a summary of Propositions 3–6, we have the following.

Theorem 7. *Let D be an asymmetric CKI-digraph with covering number 3. Then $D \cong \overrightarrow{C}_5$ or $D \cong \overrightarrow{C}_7(1, 2)$.*

Proof. We analyze all the possibilities for $D[\{u_1, v_1, w_1\}]$. If $D[\{u_1, v_1, w_1\}]$ has no arcs, then $\{u_1, v_1, w_1\}$ is a kernel of D, which is a contradiction. If $D[\{u_1, v_1, w_1\}]$ has exactly one arc, then by Proposition 6, $D \cong \overrightarrow{C}_5$ or $D \cong \overrightarrow{C}_7(1, 2)$. By Propositions 3 and 4, if $D[\{u_1, v_1, w_1\}]$ has at least two arcs, then $D[\{u_1, v_1, w_1\}]$ is a path of length two and hence, by Proposition 5, $D \cong \overrightarrow{C}_5$ or $D \cong \overrightarrow{C}_7(1, 2)$. □

As a summary of Corollaries 1 and 2(i) and Theorems 6 and 7, we have the following.

Proposition 7. *Let D be a CKI-digraph with covering number at most three.*

(i) *The covering number of Asym(D) is one if and only if* $D \cong \overrightarrow{C}_3$.
(ii) *The covering number of D is one if and only if* $D \cong \overrightarrow{C}_m(1, \pm 2, \pm 3, \ldots, \pm\lfloor \frac{m}{2} \rfloor)$.
(iii) *The covering number of Asym(D) is two if and only if* $D \cong \overrightarrow{C}_4(1, 2)$.
(iv) *The covering number of D is not equal to two.*
(v) *If D is an asymmetric digraph, then the covering number of D is three if and only if* $D \cong \overrightarrow{C}_5$ *or* $D \cong \overrightarrow{C}_7(1, 2)$.

Proposition 8. *Let D be a KP-digraph with covering number at most three.*

(i) *The covering number of D is at most two if and only if D is* $\overrightarrow{C}_3$*-free.*
(ii) *The covering number of Asym(D) is two if and only if D has no induced subdigraph isomorphic to* $\overrightarrow{C}_3$ *nor to* $\overrightarrow{C}_4(1, 2)$.
(iii) *Let D be asymmetric and* $\sigma(asym(D)) = 3$. *Then D is a KP-digraph if and only if D has no induced subdigraph isomorphic to* $\overrightarrow{C}_3$, $\overrightarrow{C}_4(1, 2)$, $\overrightarrow{C}_5$ *nor* $\overrightarrow{C}_7(1, 2)$.

5. Consequences of the results

In this section we review some previous results that can be obtained with our results in case that $\sigma(D) \leq 2$, $\sigma(Asym(D)) \leq 2$ and in case that the asymmetric digraph D has covering number three.

Theorem 8 (*Theorem 1.4 [15]*)**.** *If D is a digraph with* $\sigma(D) \leq 2$ *such that each directed cycle of length 3 has two symmetrical arcs, then D is a KP-digraph.*

Proof. Since each directed cycle of length 3 has two symmetrical arcs, then D has no induced $\overrightarrow{C}_3$ nor an induced $\overrightarrow{C}_4(1, 2)$ because $(0, 1, 2, 0)$ is directed cycle of length 3 with exactly one symmetrical arc. So, by Theorem 8(i), D is a KP-digraph. □

Hence Theorem 1.4 [15] is a consequence of Theorem 8.

Theorem 9 (*Theorem 2.3 [15]*)**.** *Let D be a digraph with* $\sigma(D) \leq 3$ *such that each directed cycle of length 3 is symmetrical. If every directed cycle of length 5 has two diagonals, then D is a KP-digraph.*

Proof. Since each directed cycle of length 3 is symmetrical, D has no induced cycles of length 3 and hence by Theorem 8(i), D is a KP-digraph in case $\sigma(D) \leq 2$. Let D be asymmetric with $\sigma(D) = 3$. Every directed cycle of length 5 has two diagonals, so D has no induced cycle of length 5. Moreover, D has no induced $\overrightarrow{C}_7(1, 2)$ because the 5-cycle $(0, 1, 2, 4, 5, 0)$ has only one diagonal, namely the arc $(0, 2)$. By Theorem 8(ii), D is a KP-digraph. □

Theorem 10 (*Theorem 2.4 [15]*)**.** *Let D be a digraph with* $\sigma(D) \leq 3$*, but without directed cycles of length 3. If every directed cycle of length 5 has two diagonals, then D is a KP-digraph.*

Proof. Analogously to the proof of Theorem 9. □

Hence, Theorems 2.3 and 2.4 [15] are both consequences of Theorem 8.

Theorem 11 (*Theorem 2.1 [14]*)**.** *Let D be a digraph such that every directed triangle has two symmetric arcs and* $\sigma(D) \leq 3$. *If each directed cycle* $\mathcal{C}$ *of length 5 in D satisfies at least one of the following properties: (a)* $\mathcal{C}$ *has two diagonals, (b)* $\mathcal{C}$ *has three symmetrical arcs, then D is a KP-digraph.*

Proof. Analogously to the proof of Theorem 9. □

Hence, Theorem 2.1 [14] is a consequence of Theorem 8.

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