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On the acyclic disconnection of multipartite tournaments[☆]

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ABSTRACT

The acyclic disconnection of a digraph D is the maximum number of components that can be obtained by deleting from D the set of arcs of an acyclic subdigraph. We give bounds for the acyclic disconnection of strongly connected bipartite tournaments and of regular bipartite tournaments. For the latter case, we exhibit an infinite family of tournaments with acyclic disconnection equal to 4.

1. Introduction

A coloring of the vertices of a digraph D is a *proper coloring* if each color is assigned to at least one vertex of D. The *acyclic disconnection* $\overrightarrow{\omega}(D)$ of a digraph D was defined in [7] as the maximum possible number of connected components of the underlying graph of $D \setminus A(D^*)$ where D^* is an acyclic subdigraph of D. Equivalently, $\overrightarrow{\omega}(D)$ can be defined as the maximum number of colors for which there is a coloring of the vertices of D not producing well-colored directed cycles (a directed cycle $\overrightarrow{C}$ is *well-colored* if every pair of adjacent vertices of $\overrightarrow{C}$ is colored with different colors), see more details in Section 2. There are other forms of defining the acyclic disconnection which are also useful depending on the context (Proposition 4). A small value of $\overrightarrow{\omega}(D)$ implies a complex pattern of cycles in D. Besides the important results for this parameter in general digraphs that were introduced in [7], the acyclic disconnection (and particularly, the so called *directed triangle free disconnection*,a very closely related notion defined in the same paper) has mainly been studied for tournaments, specially, for circulant tournaments (see for instance the already mentioned paper, [2,4]) and for some classes of (non-circulant) regular tournaments in [5].

The acyclic and the directed triangle free disconnection are also related to the dichromatic number of a digraph, a definition established by Neumann-Lara in [8]. Infinite families of regular tournaments with given fixed acyclic disconnection s and dichromatic number $r (2 \le s \le r)$ were determined in [7,5] for small values of integers s and r, namely, $2 \le s \le r \le 3$. In [6], the general problem was studied for all possible values of the parameters.

In this paper, we begin the study of the acyclic disconnection of general multipartite tournaments, in particular we focus on the class of bipartite tournaments. After some preliminaries and terminology in Section 2 (in general, we follow [1] for the basics), in Section 3 we prove that for every c-partite tournament T, in order to determine $\overrightarrow{\omega}(T)$ it is sufficient to consider directed cycles of lengths 3, 4 and 6 (Theorem 8). Evidently, directed triangles are excluded in case of bipartite tournaments. This result generalizes a previous one for tournaments (see Proposition 6.3 of [7]).

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In Section 4, we observe that $3 \leq \overrightarrow{\omega}(T) \leq n-1$ for every strongly connected bipartite tournament T of order n. In Corollary 17(ii), we show a strongly connected bipartite tournament T of order n with acyclic disconnection $n-1$. Since the acyclic disconnection of a directed cycle on 4 vertices is 3, the bounds given cannot be improved for strongly connected bipartite tournaments in general. On the other hand, if T is an r-regular bipartite tournament, we have that $\overrightarrow{\omega}(T) \leq 2r+1$ (Theorem 13). We also give examples showing that this bound is tight (Corollary 18(i)).

Finally, in Section 5 we study the acyclic disconnection of regular bipartite tournaments without concordance. If T is an r-regular bipartite tournament without concordance, our best bound for the acyclic disconnection is $3 \leq \overrightarrow{\omega}(T) \leq r+2$ (Theorem 22). In spite of the fact that we were not able to find an infinite family of regular bipartite tournaments with acyclic disconnection 3 nor $r+2$, we show such an infinite family of digraphs without concordance with acyclic disconnection 4.

2. Preliminaries and terminology

A digraph D is *asymmetric* if (v,u) is not an arc of D whenever (u,v) is an arc of D. In this paper every digraph is asymmetric, finite and without loops or multiple arcs.

Let D be a digraph. The vertex set and the arc set of D are denoted by $V(D)$ and $A(D)$, respectively. We say that a graph is *nontrivial* if it has order greater than one. For a set $X \subseteq V(D)$, we denote by $D[X]$ and $D \setminus X$ the subdigraphs of D induced by X and $V(D) \setminus X$, respectively.

Let $u \in V(D)$, the out-neighborhood and in-neighborhood of u are denoted by $N^+(u)$ and $N^-(u)$, respectively. If $S \subseteq V(G)$, the set $S \cap N^+(u)$ (resp. $S \cap N^-(u)$) is denoted by $N^+(u;S)$ (resp. $N^-(u;S)$). Analogously, $d^+(u)$, $d^+(u;S)$, $d^-(u)$ and $d^-(u;S)$ denote the out-degree, the out-degree with respect to S, the in-degree and the in-degree with respect to S of u, respectively. A vertex v is a *source* of D if $d^-(v)=0$. A digraph D is *r-regular* if $d^+(v)=d^-(v)=r$ for every vertex v of D.

We say that two distinct vertices u and v of a digraph D are *concordant vertices* (resp. *discordant vertices*) if $N^+(u)=N^+(v)$ and $N^-(u)=N^-(v)$ (resp. $N^+(u)=N^-(v)$ and $N^-(u)=N^+(v)$). If u and v are not concordant for every $u \neq v \in V(D)$, then we say that D is a *digraph without concordance*.

Throughout this article a cycle of D means a directed cycle of D. A digraph D without cycles is an *acyclic digraph*. It is well known that D is acyclic if and only if there exists a labeling of its vertex set $V(D)=\{v_0,v_1,\ldots,v_n\}$ such that v_0 is a source of D and for every $1 \leq k \leq n$, v_k is a source of $D \setminus \{v_0,v_1,\ldots,v_{k-1}\}$.

A *tournament* T is a digraph such that there is exactly one arc between every pair of vertices. Let $c \geq 2$ be an integer, a *c-partite tournament* is an orientation of a complete c-partite graph. In particular, every n-partite tournament with n vertices is a tournament. A 2-partite tournament is called a *bipartite tournament*. If T is an r-regular bipartite tournament with partite sets U and V, then $|U|=|V|=2r$.

For $m \geq 3$ we denote by $\mathbb{Z}_m$ the cyclic group of integers modulo m. Let J be a non-empty subset of $\mathbb{Z}_m \setminus \{0\}$ with the property that $-j \notin J$ whenever $j \in J$. A *circulant*(or *rotational) digraph*, denoted by $\overrightarrow{C}_m(J)$, has vertex set $V(\overrightarrow{C}_m(J))=\mathbb{Z}_m$ and arc set $A(\overrightarrow{C}_m(J))=\{(i,j) : j-i \in J\}$. Notice that $\overrightarrow{C}_m(J)$ is a vertex transitive digraph.

Let D be a digraph and $\{F_u\}_{u\in V(D)}$ be a family of digraphs. The *Zykov sum* [9] (X-*Join* [3] or *lexicographic sum*) of the digraph D with the family $\{F_u\}$, denoted by $\sigma(D,F_u)$ has vertex set $\{(u,x) : u \in V(D), x \in V(F_u)\}$ and $((u,x),(v,y))$ is an arc of $\sigma(D,F_u)$ if and only if

$$(u,v) \in A(D) \quad \text{or} \quad u=v \quad \text{and} \quad (x,y) \in A(F_u).$$

Let D and F be digraphs, the *composition* $D[F]$ (or *lexicographic product* $D \circ F$ [3]) is the digraph $\sigma(D,F_u)$ where F_u is an isomorphic copy of F for every $u \in V(D)$. It is well known that the composition of digraphs is associative but not commutative. Furthermore the composition of two regular digraphs is again a regular digraph.

For a complete graph K_n, let $\overline{K}_n$ denote the complement of K_n.

Proposition 1. *Let D be a nontrivial digraph. A Zykov sum $\sigma(D,F_u)$ is a bipartite tournament if and only if D is a bipartite tournament and F_u is isomorphic to a graph $\overline{K}_{n_u}$, with $n_u \geq 1$, for every $u \in V(D)$.*

Proof. Suppose that $\sigma(D,F_u)$ is a bipartite tournament. By definition, any Zykov sum $\sigma(D,F_u)$ has an induced subdigraph D' isomorphic to D. If D has no arcs, then all arcs of $\sigma(D,F_u)$ are of the form $((u,x),(u,y))$ where (x,y) is an arc of F_u and this is not possible since $\sigma(D,F_u)$ is a bipartite tournament and D has at least two vertices. Therefore there is an arc (u,v) of D and, consequently, an arc $((u,x),(v,y))$ of D'. This implies that D' is an induced subdigraph of a bipartite tournament with at least one edge, therefore, D' and also D are bipartite tournaments too. Moreover, $N^+(u) \cup N^-(u) \neq \emptyset$ for every $u \in V(D)$, since D is a nontrivial bipartite tournament. If F_u has an arc xy and $v \in N^+(u) \cup N^-(u) \neq \emptyset$ for some $u \in V(D)$, then $\{(u,x),(u,y),(v,z)\}$ induces a (not directed) triangle of $\sigma(D,F_u)$, where z is any vertex of F_v. We reach a contradiction since $\sigma(D,F_u)$ is bipartite, therefore $F_u \cong \overline{K}_{n_u}$ for every $u \in V(D)$.

On the other hand, if D is a bipartite tournament with partite sets U and V, then $\sigma(D,\overline{K}_{n_u})$ is a bipartite tournament with bipartition (U',V'), where $U'=\{(u,x) : u \in U,\ x \in V(\overline{K}_{n_u})\}$ and $V'=\{(v,y) : v \in V,\ y \in \overline{K}_{n_v}\}$. □

Corollary 2. *Let D be a nontrivial digraph. A composition $D[F]$ is a (regular) bipartite tournament if and only if D is a (regular) bipartite tournament and $F \cong \overline{K}_n$.*

Remark 3. A bipartite tournament T has concordant vertices if and only if $T = \sigma(D, F_u)$ and F_v is nontrivial for some $v \in V(D)$.

Proof. Let $v \in V(D)$ be such that F_v has at least two vertices. If $T = \sigma(D, \overline{K}_{n_u})$ and $x_1 \neq x_2 \in V(K_{n_v})$, it is easy to see that (v, x_1), (v, x_2) are concordant. Now, let T be a bipartite tournament, $x, y \in V(T)$ concordant vertices of T, and $T' = T \setminus \{x\}$. Then, T is isomorphic to the Zykov sum $\sigma(T', F_u)$, where $V(F_y) = \{x, y\}$ and $V(F_u) = \{u\}$ for every $u \neq y$. □

We denote by $\omega(D)$ the number of connected components of the underlying graph of D. A spanning subdigraph D_0 of D is called a *linear* $\overrightarrow{C}$*-transversal* of D if $A(D_0)$ contains at least one arc of every cycle of D. The set of linear $\overrightarrow{C}$-transversals of D is denoted by $Tr(D)$. Let π be a partition of $V(D)$, an arc uv is external if u and v are elements of different parts of π. A partition of the vertex set of a digraph D is an *externally acyclic partition* if the digraph induced by the external arcs of D is acyclic.

The following proposition gives different characterizations of $\overrightarrow{\omega}(D)$.

Proposition 4 (*Proposition 2.2 [7]*)**.** *Each of the following values is equal to* $\overrightarrow{\omega}(D)$.

(i) $\max\{\omega(D \setminus F) : F \subseteq A(D), F \text{ acyclic}\}$.
(ii) $\max\{\omega(W) : W \in Tr(D)\}$.
(iii) *The maximum cardinality of an externally acyclic partition of D.*
(iv) *The maximum number of colors in a proper coloring of $V(D)$ not producing well-colored cycles.*

For $s \geq 1$, let Γ_s denote the set of colors $\{c_1, c_2, \ldots, c_s\}$. If D is a digraph and $\varphi : V(D) \to \Gamma_s$ is a vertex coloring of D, then the coloring φ induces two natural spanning subdigraphs of D. The *monochromatic digraph* of D, $M_\varphi(D)$, is the spanning subdigraph of D with arc set $\{uv \in A(D) : \varphi(u) = \varphi(v)\}$ and the *heterochromatic digraph* of D, $H_\varphi(D)$, is the spanning subdigraph of D with arc set $\{uv \in A(D) : \varphi(u) \neq \varphi(v)\}$. We say that a coloring $\varphi : V(D) \to \Gamma_s$ is *externally acyclic* if $H_\varphi(D)$ is an acyclic digraph.

Proposition 5. *If $\overrightarrow{\omega}(D) \geq k$, then there is a proper coloring γ of D with exactly k colors such that $H_\gamma(D)$ contains no cycles.*

Proof. If $\overrightarrow{\omega}(D) = t \geq k$, then there is a proper coloring δ with t colors $\{c_1, c_2, \ldots, c_t\}$ such that $H_\delta(D)$ is acyclic. Let γ be the proper coloring of D with k colors obtained from δ as follows:

$$\gamma(v) = \begin{Bmatrix} c_i & \text{if } \delta(v) = c_i \text{ with } 1 \leq i \leq k-1 \\ c_k & \text{if } \delta(v) = c_i \text{ with } i \geq k. \end{Bmatrix}$$

If $\gamma(u) \neq \gamma(v)$, then $\delta(u) \neq \delta(v)$; therefore $H_\gamma(D)$ is a subdigraph of $H_\delta(D)$. This implies that $H_\gamma(D)$ is also acyclic. □

The following proposition is a reformulation of Proposition 4(iv).

Proposition 6. *For any digraph D, $\overrightarrow{\omega}(D)$ is the largest integer n for which there is an externally acyclic coloring $\varphi : V(D) \longrightarrow \Gamma_n$.*

Let φ be a coloring of a digraph D, we say that color c is a *singular class* of φ if $|\varphi^{-1}(\{c\})| = 1$. Let $S \subset V(D)$, we denote by $\varphi(S)$ the set $\{\varphi(u) : u \in S\}$.

3. Forbidden cycles in an externally acyclic coloring

Let $c \geq 2$. We will prove that if T is a c-partite tournament and φ is a coloring of T such that $H_\varphi(T)$ has a cycle of any length, then $H_\varphi(T)$ has also a cycle of length 3, 4 or 6. Let $\overrightarrow{C}$ be a cycle of a digraph D and $u, v \in V(\overrightarrow{C})$. We say that $\{u, v\}$ is a *chord* of $\overrightarrow{C}$ if uv or vu is a chord of $\overrightarrow{C}$.

Remark 7. Let T be a c-partite tournament, $\varphi : V(T) \to \Gamma_s$ be a coloring which is not externally acyclic and $\overrightarrow{C}$ be a cycle of minimum length in $H_\varphi(T)$. If $\{u, v\}$ is a chord of $\overrightarrow{C}$, then $\varphi(u) = \varphi(v)$.

Theorem 8. *Let T be a c-partite tournament and φ be a vertex coloring of T. If φ is not an externally acyclic coloring, then $H_\varphi(T)$ has a 3-cycle, a 4-cycle or a 6-cycle.*

Proof. Let T be a c-partite tournament with partite sets $V_1, V_2, \ldots, V_c$ and φ be a vertex coloring of T that is not externally acyclic. Suppose by contradiction that $\overrightarrow{C} = (v_0, v_1, \ldots v_{t-1}, v_0)$ is a cycle of minimum length in $H_\varphi(T)$ with $t \neq 3, 4, 6$.

Claim 1. *If $v_i, v_j \in V(\overrightarrow{C})$ and $j \neq i+1, i, i-1, i-2$, then either $\{v_i, v_j\}$ or $\{v_i, v_{j+1}\}$ is a chord of $\overrightarrow{C}$, but not both.*

Proof of Claim. If $\{v_i, v_j\}$ and $\{v_i, v_{j+1}\}$ are not chords of $\overrightarrow{C}$, then v_i, v_j and v_{j+1} are in the same partite set of T, which is a contradiction because $(v_j, v_{j+1}) \in A(T)$. On the other hand, if $\{v_i, v_j\}$ and $\{v_i, v_{j+1}\}$ are both chords of $\overrightarrow{C}$, then by Remark 7, $\varphi(v_j) = \varphi(v_i) = \varphi(v_{j+1})$. This is impossible because $\overrightarrow{C}$ is a cycle of $H_\varphi(T)$. □

Case 1. $t = 2s + 1$ for some $s \geq 2$.

Since $\overrightarrow{C}$ is a cycle of minimum length, by Claim 1, the chords of type $\{v_i, v_{i+s}\}$ and $\{v_i, v_{i+s+1}\}$ induce a perfect matching of the vertices of $\overrightarrow{C}$ which is a contradiction because t is odd.

Case 2. $t = 2s$ for some $s \geq 4$.

By Claim 1, we can choose $s \leq i \leq s + 1$ such that $\{v_0, v_i\}$ is a chord of $\overrightarrow{C}$ and $\{v_0, v_{i+1}\}$ and $\{v_0, v_{i-1}\}$ are not chords of $\overrightarrow{C}$. Thus there exists a partite set U of T such that $v_0, v_{i-1}, v_{i+1} \in U$. Analogously, there exists a partite set V of T such that $v_1, v_i, v_{2s-1} \in V$. Notice that $v_2 \notin V$ since $v_1 \in V$ and (v_1, v_2) is an arc of T. This implies that $\{v_2, v_i\}$ and $\{v_2, v_{2s-1}\}$ are also chords of $\overrightarrow{C}$. By Remark 7, $\varphi(v_0) = \varphi(v_i) = \varphi(v_2) = \varphi(v_{2s-1})$ which is a contradiction because (v_{2s-1}, v_0) is an arc of $\overrightarrow{C}$ which is a cycle of $H_\varphi(T)$. $\square$

In the case where T is a bipartite tournament, T has no odd cycles. So we have the following corollary.

Corollary 9. *Let T be a bipartite tournament and φ be a vertex coloring of T. If φ is not an externally acyclic coloring, then $H_\varphi(T)$ has a 4-cycle or a 6-cycle.*

The following example shows that there exists a vertex coloring φ with 3 colors of a bipartite tournament T such that $H_\varphi(T)$ has a 6-cycle but no 4-cycle. This example and Corollary 9 show that it is not enough to check that $H_\varphi(T)$ does not have cycles of length 4 to assure that $H_\varphi(T)$ is acyclic.

Example 10. Let T be a bipartite tournament with partite sets $U = \{v_0, v_2, v_4\}$ and $V = \{v_1, v_3, v_5\}$ in which $\overrightarrow{C} = (v_0, v_1, v_2, v_3, v_4, v_5, v_0)$ is a hamiltonian cycle. Let $\varphi : V(T) \to \Gamma_3$ be the vertex coloring defined by:

$$\varphi(v_i) = \varphi(v_{i+3}) = c_i, \quad \text{for } i = 0, 1, 2.$$

Clearly $\overrightarrow{C}$ is a cycle of $H_\varphi(T)$. Note that every 4-cycle in T has an arc of type either (v_i, v_{i+3}) or (v_{i+3}, v_i) for some $i = 0, 1, 2$. Hence, $H_\varphi(T)$ has no 4-cycles.

$$\begin{array}{ccccc} c_0 & \to & c_1 & \to & c_2 \\ \uparrow & & & & \downarrow \\ c_2 & \leftarrow & c_1 & \leftarrow & c_0 \end{array}$$

Note that if T is a tournament and φ is a vertex coloring of T such that $H_\varphi(T)$ has a 6-cycle, then $H_\varphi(T)$ has a 3-cycle or a 4-cycle. So, we have the following.

Corollary 11 (*Proposition 6.3 [7]*)**.** *Let T be a tournament and φ be a vertex coloring of T. If φ is not an externally acyclic coloring, then $H_\varphi(T)$ has a 3-cycle or a 4-cycle.*

4. Bipartite tournaments

In this section, we find bounds on the acyclic disconnection of bipartite tournaments. If T is an acyclic bipartite tournament, then $\overrightarrow{\omega}(T) = n$, where $n = |V(T)|$. Throughout this paper we consider bipartite tournaments with at least one cycle. If T is not acyclic and U and V are the parts of T, then $|U|, |V| \geq 2$.

Proposition 12. *Let T be a bipartite tournament of order n. If T is not acyclic, then $3 \leq \overrightarrow{\omega}(T) \leq n - 1$.*

Proof. Let u and v be vertices in different partite sets of T and define $\varphi : V(T) \to \Gamma_3$ as follows: $\varphi(u) = 1$, $\varphi(v) = 2$ and $\varphi(x) = 3$ for each $x \in V(T) \setminus \{u, v\}$. Since each cycle of T contains adjacent vertices with color 3, φ is externally acyclic. $\square$

Theorem 13. *If T is an r-regular bipartite tournament of order $4r$, then $\overrightarrow{\omega}(T) \leq 2r + 1$.*

Proof. Let U and W be the partite sets of T. Suppose by contradiction that $\varphi : V(T) \to \Gamma_k$ is an externally acyclic coloring of T with $k \geq 2r + 2$ colors.

Since $H_\varphi(T)$ is acyclic, we may assume that there exists a source $u \in U$ of $H_\varphi(T)$. That is, $\varphi(N^-(u)) = \{\varphi(u)\}$. Let $S \subset W$ be such that:

$$S \text{ has exactly one vertex of each color in } \varphi(W) \tag{1}$$

Let A be the set of vertices in W with color $\varphi(u)$. Since $N^-(u) \subseteq A$ and T is an r-regular bipartite tournament, $|A| \geq r$. By (1), $|S \cap A| = 1$ and therefore

$$|\varphi(W)| = |S| = |S \cap A| + |S \cap (W \setminus A)| \leq 1 + |W \setminus A| \leq 1 + 2r - r = r + 1. \tag{2}$$

Notice that

$$\Gamma_k \setminus \varphi(W) \subset \varphi(U). \tag{3}$$

Let $S' \subset U$ be such that

$$S' \text{ has exactly one vertex of each color in } \Gamma_k \setminus \varphi(W). \tag{4}$$

Since T is colored by at least $2r+2$ colors, (2) and (3), we have that

$$|S'| = |\Gamma_k \setminus \varphi(W)| \geq k - (r+1) \geq 2r + 2 - r - 1 = r + 1. \tag{5}$$

Let $v \in S$. Since T is r-regular, and (5), there exist $u_1, u_2 \in S'$ such that $u_1 \in N^+(v)$ and $u_2 \in N^-(v)$. By (3) and (4), $\varphi(u_1) \neq \varphi(v) \neq \varphi(u_2)$. Since $v \in N^-(u_1) \cap N^+(u_2)$ and T is an r-regular tournament, there exists $v^* \in N^+(u_1) \cap N^-(u_2)$. By (3) and (4), $\varphi(u_1) \neq \varphi(v^*) \neq \varphi(u_2)$. Therefore (v, u_1, v^*, u_2, v) is a cycle of $H_\varphi(T)$ which is a contradiction. □

For a digraph W, we denote by W^0 the digraph W without its isolated vertices.

We will use the following result to prove that the bounds of Proposition 12 are tight for strongly connected bipartite tournaments, and that the upper bound of Theorem 13 is tight for regular bipartite tournaments.

Theorem 14 (*Theorem 3.4 [7]*)**.** *Let $\sigma(D, F_u)$ be a digraph. Then*

$$\overrightarrow{\omega}(\sigma(D, F_u)) = \max_{W \in Tr(D)} \left\{ \omega(W^0) + \sum_{u \notin V(W^0)} |F_u| \right\}.$$

As a consequence of Theorem 14 and the definitions of Zykov sum and composition, we have the following two corollaries for bipartite tournaments.

Corollary 15. *Let $\sigma(D, \overline{K}_{n_u})$ be a bipartite tournament, where $n_u \geq 1$. Then*

$$\overrightarrow{\omega}(\sigma(D, \overline{K}_{n_u})) = \max_{W \in Tr(D)} \left\{ \omega(W^0) + \sum_{u \notin V(W^0)} |n_u| \right\}.$$

Corollary 16. *Let $D[\overline{K}_n]$ be a bipartite tournament. Then*

$$\overrightarrow{\omega}(D[\overline{K}_n]) = \max_{W \in Tr(D)} \{\omega(W^0) + n|V(D) \setminus V(W^0)|\}.$$

Corollary 17. *Let $T = \sigma(\overrightarrow{C}_4, \overline{K}_{n_i})$ $(0 \leq i \leq 3)$, where $\overrightarrow{C}_4 = (u_0, u_1, u_2, u_3, u_0)$, $n_2 \geq n_0$ and $n_3 \geq n_1$. Then $\overrightarrow{\omega}(\sigma(\overrightarrow{C}_4, \overline{K}_{n_i})) = n_2 + n_3 + 1$. Moreover,*

(i) *For every $r \in \mathbb{N}$, there exists an r-regular bipartite tournament T_r such that $\overrightarrow{\omega}(T_r) = 2r + 1$.*
(ii) *For every $n, k \in \mathbb{N}$ such that $n \geq 4$ and $\lceil n/2 \rceil + 1 \leq k \leq n - 1$, there exists a strongly connected bipartite tournament T of order n such that $\overrightarrow{\omega}(T) = k$.*

Proof. We may assume that $U = \{\overline{K}_{n_0}, \overline{K}_{n_2}\}$ and $V = \{\overline{K}_{n_1}, \overline{K}_{n_3}\}$ are the partite sets of T.

Let $S \in Tr(\overrightarrow{C}_4)$. We define $\rho(S) = \omega(S^0) + \sum_{u_i \notin V(S^0)} (n_{u_i})$.

If $|A(S^0)| \geq 2$, then $\rho(S) \leq 2 + \max\{n_2, n_3\}$ and if $|A(S^0)| = 1$, then $\rho(S) \leq 1 + n_2 + n_3$. By Theorem 14,

$$\overrightarrow{\omega}(T) = \max_{S \in Tr(\overrightarrow{C}_4)} \rho(S) \leq 1 + n_2 + n_3.$$

Let $W \in Tr(\overrightarrow{C}_4)$ such that $A(W^0) = \{u_0u_1\}$. Note that $\rho(W) = 1 + n_2 + n_3$. Hence,

$$\overrightarrow{\omega}(T) = 1 + n_2 + n_3.$$

(i) Observe that $T_r = \overrightarrow{C}_4[K_r]$ is such that $\overrightarrow{\omega}(T_r) = 2r + 1$.
(ii) Let $T = \sigma(\overrightarrow{C}_4, \overline{K}_{n_i})$ $(0 \leq i \leq 3)$, where $\overrightarrow{C}_4 = (u_0, u_1, u_2, u_3, u_0)$, $n_2 \geq n_0$ and $n_3 \geq n_1$. If k and n are integers such that $\lceil n/2 \rceil + 1 \leq k \leq n - 1$ and $n_2 + n_3 = k - 1$, then by (i) it follows that $\overrightarrow{\omega}(T) = k$. □

By Corollary 17(ii) and since $\overrightarrow{\omega}(\overrightarrow{C}_4) = 3$, the bounds given in Proposition 12 are best possible for strongly connected bipartite tournaments. Corollary 17 (i) shows that the upper bound of Theorem 13 is also tight for regular bipartite tournaments.

The following corollary gives lower bounds on the acyclic disconnection of bipartite tournaments with concordance.

Corollary 18. *Let T be a bipartite tournament with partite sets U and V. If T is not acyclic, then we have the following bounds on the acyclic disconnection of T.*

(i) *If $U' \subseteq U$ and $V' \subseteq V$ such that $|U'| = s$ and $|V'| = t$, and the vertices of U' (resp. of V') are concordant, then $\overrightarrow{\omega}(T) \geq s + t + 1$.*
(ii) *If T has concordant vertices, then $\overrightarrow{\omega}(T) \geq 4$.*
(iii) *If $T \cong D[\overline{K}_r]$, then $\overrightarrow{\omega}(T) \geq 2r + 1$.*

Proof. (i) Let $u' \in U'$, $v' \in V'$, $W = (U' \cup V') \setminus \{u', v'\}$ and $T' = T \setminus W$. If $V(F_u) = \{u\}$ whenever $u \in V(T') \setminus \{u', v'\}$, $V(F_{u'}) = U'$ and $V(F_{v'}) = V'$, then it is clear that $T' \cong \sigma(T', F_u)$. Thus, by Corollary 15, $\overrightarrow{\omega}(T) \geq s + t + 1$.

(ii) and (iii) follow by Remark 3 and (i). □

Notice that if T is a bipartite tournament of order n with concordance, then $4 \leq \overrightarrow{\omega}(T) \leq n - 1$ (Corollary 18(ii) and Proposition 12). Finally, by Corollary 17 (ii), the previous bounds are tight for bipartite tournament with concordance.

The results proved in this section show that if a bipartite tournament T has concordant vertices, then $\overrightarrow{\omega}(T)$ is relatively big with respect to its order. Furthermore, in order to find bipartite tournaments with minimal values for their acyclic disconnection, the most interesting cases are the highly "cyclic" regular bipartite tournaments without concordance.

5. Bipartite regular tournaments without concordance

In this section we only consider regular bipartite tournaments without concordance.

Lemma 19. *Let T be a regular bipartite tournament without concordance, U and V be the partite sets of T. If $\varphi : V(T) \to \Gamma_s$ is a vertex coloring of T, then*

(i) *For every pair $u \neq v \in U$ (resp. $u \neq v \in V$), there exists a 4-cycle containing u and v.*
(ii) *If the set U (resp. V) has two vertices with colors which are not in $\varphi(V)$ (resp. in $\varphi(U)$), then $H_\varphi(T)$ has a 4-cycle.*

Proof. (i) Since u and v are not concordant vertices, there exist $x \in N^+(u) \cap N^-(v)$ and $y \in N^+(v) \cap N^-(u)$. Therefore (u, x, v, y, u) is a 4-cycle of T. (ii) follows from (i). □

Lemma 20. *Let T be a regular bipartite tournament without concordance, U and V the partite sets of T. If $\varphi : V(T) \to \Gamma_s$ is an externally acyclic vertex coloring of T, then*

(i) *There are at most two singular classes and $|\varphi(V) \cap \varphi(U)| \geq s - 2$.*
(ii) *Each element c_i of $\varphi(U) \setminus \varphi(V)$ is a singular class of φ.*

Proof. As a consequence of Lemma 19(ii), there is at most one color in U not being used in V and reciprocally; so (i) is settled. Finally, (ii) is proved by pointing out that if there exists a color in U not being used in V, then this color must be a singular class by Lemma 19(ii). □

By Lemma 20(i), if T has no concordant vertices, then an externally acyclic partition of $V(T)$ with $\overrightarrow{\omega}(T)$ classes has at most two singular classes. Then we have the following corollary to Theorem 14.

Corollary 21. *Let D be a bipartite tournament without concordance. Then*

$$\overrightarrow{\omega}(D[\overline{K}_n]) \leq \max_{W \in Tr(D)} \{\omega(W^0) + 2n\}.$$

Proof. Note that $|V(D) \setminus V(W^0)| \leq 2$, since there are at most two singular classes of an externally acyclic coloring, by Lemma 20(i). □

Theorem 22. *If T is an r-regular bipartite tournament without concordance, then $\overrightarrow{\omega}(T) \leq r + 2$.*

Proof. This proof is analogous to the proof of Theorem 13. Using the same notations of the proof of Theorem 13, we obtain $|S| \leq r + 1$ and $|S'| \geq 2$. This is a contradiction to the fact that U has at most one vertex s such that $\varphi(s) \notin \varphi(W)$, by Lemma 19(ii). □

We denote by B_k the circulant bipartite tournament $\overrightarrow{C}_{4k}(\{1, 3, \ldots, 2k - 1\})$. Our last result states that the digraphs B_k with $k \geq 2$ form an infinite family of regular bipartite tournaments without concordance with acyclic disconnection 4. To show that we need the following lemmas.

Lemma 23. $\overrightarrow{\omega}(B_2) = 4$.

Proof. Let φ be an externally acyclic coloring with at least five colors. Since $H_\varphi(B_2)$ is acyclic and B_2 is vertex transitive, then we may assume that 0 is the source of $H_\varphi(B_2)$ and that $\varphi(0) = R$. Since $7, 5 \in N^-(0)$, we have that $\varphi(\{7, 5\}) = R$. Notice that the rest of the vertices of B_2 (five in total) are colored with at least four colors. Therefore, there are at least three singular classes which is a contradiction by Lemma 20 (i).

Let $\varphi : V(B_2) \to \{R, B, G, Y\}$ be the coloring such that $\varphi(\{0, 2, 5, 7\}) = R$, $\varphi(\{1, 6\}) = B$, $\varphi(4) = G$ and $\varphi(3) = Y$. It is clear that φ is an externally acyclic coloring of B_2. □

Lemma 24. *Let $k \geq 2$. For each $i = 0, 1, \ldots, 2k-1$,*

(i) *the vertices i and $i + 2k$ are discordant in B_k.*
(ii) *the set $\{i, i+1, 2k+i, 2k+i+1\}$ induces a 4-cycle in B_k and*
(iii) *the digraph $B_k \setminus \{i, i+1, i+2k, i+2k+1\}$ is isomorphic to B_{k-1}.*

Proof. (i) and (ii) are easy consequences of the definition of B_k. Since B_k is a vertex transitive bipartite tournament and the fact that for each i, the map $\phi : j \to j + i$, is an automorphism of B_k, we may assume that $i = 2k - 2$. Then (iii) follows since

$$\psi(j) = \begin{cases} j & \text{if } 0 \leq j < 2k-2, \\ j-2 & \text{if } 2k \leq j < 4k-2. \end{cases}$$

shows that $B_k \setminus \{i, i+1, i+2k, i+2k+1\}$ is isomorphic to B_{k-1} in the case where $i = 2k-2$. □

Theorem 25. *If $k \geq 2$, then $\overrightarrow{\omega}(B_k) = 4$.*

Proof. Let $k \geq 2$. To prove $\overrightarrow{\omega}(B_k) \geq 4$ we define a vertex coloring $\varphi : V(B_k) \to \{R, B, G, Y\}$ as follows:

$$\varphi(2i) = \begin{cases} R & \text{if } 0 \leq 2i \leq 2k-2, \\ G & \text{if } 2i = 2k, \\ B & \text{if } 2k+2 \leq 2i \leq 4k-2, \end{cases} \tag{6}$$

and

$$\varphi(2i-1) = \begin{cases} B & \text{if } 1 \leq 2i-1 \leq 2k-3, \\ Y & \text{if } 2i-1 = 2k-1, \\ R & \text{if } 2k+1 \leq 2i-1 \leq 4k-1 \end{cases} \tag{7}$$

and show that φ is an externally acyclic coloring.

Observe that $N^-(0) = \{2i-1 : 2k+1 \leq 2i-1 \leq 4k-1\}$. By (6) and (7), $\varphi(v) = \varphi(0) = R$ for every $v \in N^-(0)$. Therefore 0 is a source of $H_\varphi(B_k)$. Analogously, 1 is a source of $H_\varphi(B_k) \setminus \{0\}$, since $\varphi(v) = \varphi(1) = B$ for every $v \in N^-(1; B_k \setminus \{0\}) = \{2i : 2k+2 \leq 2i \leq 4k-2\}$.

If $2 \leq 2j \leq 2k-2$, then

$$N^-(2j; B_k \setminus \{2j-1, 2j-2, \ldots, 0\}) \subseteq \{2i-1 : 2k+1 \leq 2i-1 \leq 4k-1\}.$$

By (6) and (7), $\varphi(u) = \varphi(2j) = R$ for each $u \in \{2i-1 : 2k+1 \leq 2i-1 \leq 4k-1\}$. Therefore $2j$ is a source of $H_\varphi(B_k) \setminus \{2j-1, 2j-2, \ldots, 0\}$.

If $2k \leq 2j \leq 4k-2$, then $N^-(2j; B_k \setminus \{2j-1, 2j-2, \ldots, 0\}) = \emptyset$ and $2j$ is also a source of $H_\varphi(B_k) \setminus \{2j-1, 2j-2, \ldots, 0\}$.

Analogously, if $3 \leq 2j-1 \leq 2k-3$, then

$$N^-(2j-1; B_k \setminus \{2j-2, 2j-3, \ldots, 0\}) \subseteq \{2i : 2k+2 \leq 2i \leq 4k-2\}.$$

Again by (6) and (7), $\varphi(v) = \varphi(2j-1) = B$ for each $v \in \{2i : 2k+2 \leq 2i \leq 4k-2\}$. Therefore $2j-1$ is a source of $H_\varphi(B_k) \setminus \{2j-2, 2j-3, \ldots, 0\}$.

Finally, if $2k-1 \leq 2j-1 \leq 4k-1$, then $N^-(2j; B_k \setminus \{2j-2, 2j-3, \ldots, 0\}) = \emptyset$ and $2j-1$ is also a source of $H_\varphi(B_k) \setminus \{2j-2, 2j-3, \ldots, 0\}$.

Therefore $H_\varphi(B_k)$ is an acyclic digraph which shows $\overrightarrow{\omega}(B_k) \geq 4$.

We proceed to prove $\overrightarrow{\omega}(B_t) \leq 4$ for $t \geq 2$. Suppose by contradiction that there is an integer $r \geq 2$ such that $\overrightarrow{\omega}(B_r) \geq 5$ and let k be the smallest such integer. By Proposition 5 there is a coloring φ of B_k with exactly 5 colors such that $H_\varphi(B_k)$ contains no cycles. Notice that $k \geq 3$ as $\overrightarrow{\omega}(B_2) = 4$ by Lemma 23.

For $m = 0, 1, \ldots, 2k-1$, let $S_m = \{m, m+1, 2k+m, 2k+m+1\}$. By Lemma 24, $B_k \setminus S_m$ is isomorphic to B_{k-1} and by the choice of k, $\overrightarrow{\omega}(B_k \setminus S_m) = \overrightarrow{\omega}(B_{k-1}) \leq 4$. Since $H_\varphi(B_k)$ contains no cycles, $H_\varphi(B_k) \setminus S_m$ must also by acyclic. Therefore, for $m = 0, 1, \ldots, 2k-1$, there is a color c_m of φ such that $\varphi(u) \neq c_m$ for each vertex u of $B_k \setminus S_m$.

Notice that $S_0, S_2, \ldots, S_{2r-2}$ are pairwise disjoint and therefore $c_{2i} \neq c_{2j}$ if $i \neq j$. Analogously $c_{2i+1} \neq c_{2j+1}$ if $i \neq j$. Moreover, if $c_i = c_j$, then $S_i \cap S_j \neq \emptyset$ and therefore $j = i-1$ or $j = i+1$ modulo $2k$.

For each $m = 0, 1, \ldots, 2k-1$, choose a vertex $s_m \in S_m$ such that $\varphi(s_m) = c_m$.

Claim 2. *If $c_i = c_{i+1}$, then color $c_i = c_{i+1}$ is a singular class of φ and $s_i = s_{i+1} = i+1$ or $s_i = s_{i+1} = i+1+2k$.*

Proof of Claim. If $c_i = c_{i+1}$, then $\{s_i, s_{i+1}\} \subset S_i \cap S_{i+1} = \{i+1, i+1+2k\}$. In any case, if color $c_i = c_{i+1}$ is not a singular class of φ, then $\varphi(i+1) = \varphi(i+1+2k)$ and $\{i+1, i+2, i+1+2k, i+2+2k\}$ induces a cycle of $H_\varphi(B_k)$ which is not possible.

If $s_i = i+1$ and $s_{i+1} = i+1+2k$ or $s_i = i+1+2k$ and $s_{i+1} = i+1$, then $\varphi(i+1) = \varphi(i+1+2k)$ which again is not possible. Therefore either $s_i = s_{i+1} = i+1$ or $s_i = s_{i+1} = i+1+2k$. □

By Lemma 20, φ has at most two singular classes. Therefore Claim 2 implies that there are at most two integers i and j such that $c_i = c_{i+1}$ and $c_j = c_{j+1}$. This, together with the fact that there are only five colors available for $s_0, s_1, \dots, s_{2k-1}$, gives $k \leq 3$. Since $k \geq 3$, the only remaining case is that of B_3.

As φ is a coloring of B_3 with exactly five colors, there are at least two colors c_i and c_{i+1} such that $c_i = c_{i+1}$; without loss of generality we may assume $i = 0$. Again by Claim 2, color $c_0 = c_1$ is a singular class of φ and either $s_0 = s_1 = 1$ or $s_0 = s_1 = 7$. Since B_3 is vertex transitive, we assume that $s_0 = s_1 = 1$.

Suppose $c_0 = c_1, c_2, c_3, c_4$ and c_5 are the five colors in φ. In this case we reach a contradiction since $\varphi(7) \neq c_i$ for $i = 0, 1, 2, 3, 4, 5$ because color $c_0 = c_1 = \varphi(1)$ is a singular class of φ and $7 \notin S_i$ for $i = 2, 3, 4, 5$.

Therefore, there is an integer $0 < t < 5$ such that $c_t = c_{t+1}$. By Claim 2, color $c_t = c_{t+1}$ is a singular class and either $s_t = s_{t+1} = t+1$ or $s_t = s_{t+1} = t+7$.

If t is even and $s_t = s_{t+1} = t+1$, then $\{1, 2, t+1, 8\}$ induces a cycle in $H_\varphi(B_3)$ and if t is even and $s_t = s_{t+1} = t+7$, then $\{1, 6, t+7, 0\}$ induces a cycle in $H_\varphi(B_3)$. In both cases we reach a contradiction.

If $t = 1$, then $c_0 = c_1 = c_2$ which is not possible since $S_0 \cap S_2 = \emptyset$. Therefore $t = 3$ in which case color $c_3 = c_4$ is a singular class and either $s_3 = s_4 = 4$ or $s_3 = s_4 = 10$. We show that in both cases $H_\varphi(B_3)$ contains a cycle.

Observe that there is a color x in φ such that $x \neq c_i$ for $i = 0, 1, \dots, 5$ because φ is a 5-coloring of B_3 with $c_0 = c_1$ and $c_3 = c_4$.

Case 1.- $s_0 = s_1 = 1$ and $s_3 = s_4 = 4$.

Since $7 \notin S_i$ for $i = 2, 3, 4, 5$ and color $c_0 = c_1$ is a singular class of φ, then $\varphi(7) = x$; analogously $\varphi(10) = x$. As none of the sets $\{7, 0, 1, 4\}$, $\{7, 8, 1, 4\}$, $\{5, 10, 1, 4\}$ or $\{9, 10, 1, 4\}$ induces a cycle of $H_\varphi(B_3)$, $\varphi(0) = x$, $\varphi(8) = x$, $\varphi(5) = x$ and $\varphi(9) = x$. Since color c_2 is not a singular class and $\varphi(8) = \varphi(9) = x$, then $\varphi(2) = \varphi(3) = c_2$. Finally, $\varphi(6) = \varphi(11) = c_5$, because color c_5 is not a singular class and $\varphi(0) = \varphi(5) = x$. In this case $\{2, 5, 6, 9\}$ induces a cycle of $H_\varphi(B_3)$.

Case 2.- $s_0 = s_1 = 1$ and $s_3 = s_4 = 10$.

Since $7 \notin S_i$ for $i = 2, 3, 4, 5$ and color $c_0 = c_1$ is a singular class of φ, then $\varphi(7) = x$; analogously $\varphi(4) = x$. As none of the sets $\{2, 7, 10, 1\}$, $\{6, 7, 10, 1\}$, $\{4, 5, 10, 1\}$ or $\{4, 9, 10, 1\}$ induces a cycle of $H_\varphi(B_3)$, $\varphi(2) = x$, $\varphi(6) = x$, $\varphi(5) = x$ and $\varphi(9) = x$. Since color c_5 is not a singular class and $\varphi(5) = \varphi(6) = x$, then $\varphi(11) = \varphi(0) = c_5$. Finally, $\varphi(3) = \varphi(8) = c_2$, because color c_2 is not a singular class and $\varphi(2) = \varphi(9) = x$. In this case $\{0, 5, 8, 9\}$ induces a cycle of $H_\varphi(B_3)$.

In all cases we reach a contradiction by finding a cycle in $H_\varphi(B_3)$. Therefore $\overrightarrow{\omega}(B_k) \leq 4$, for every $k \geq 2$. □

As a final remark we present the following intriguing conjecture.

Conjecture 26. *Let T be a bipartite tournament. Then $\overrightarrow{\omega}(T) = 3$ if and only if $T \cong \overrightarrow{C}_4$.*

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References

[1] J. Bang-Jensen, G. Gutin, Digraphs, in: Theory, Algorithms and Applications, second ed., in: Springer Monographs in Mathematics, Springer-Verlag London, Ltd., London, 2009.
[2] H. Galeana-Sánchez, V. Neumann-Lara, A class of tight circulant tournaments, Discuss. Math. Graph Theory 20 (2000) 109–128.
[3] W. Imrich, S. Klavžar, Product graphs, in: Structure and Recognition, in: Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, New York, 2000.
[4] B. Llano, V. Neumann-Lara, Circulant tournaments of prime order are tight, Discrete Math. 308 (2008) 6056–6063.
[5] B. Llano, M. Olsen, Infinite families of tight regular tournaments, Discuss. Math. Graph Theory 27 (2007) 299–311.
[6] B. Llano, M. Olsen, On a conjecture of Víctor Neumann-Lara, Electron. Notes Discrete Math. 30 (2008) 207–212.
[7] V. Neumann-Lara, The acyclic disconnection of a digraph, Discrete Math. 197/198 (1999) 617–632.
[8] V. Neumann-Lara, The dichromatic number of a digraph, J. Combin. Theory Ser. B 33 (1982) 265–270.
[9] V. Neumann-Lara, Dichromatic number, circulant tournaments and Zykov sums of digraphs, Discuss. Math. Graph Theory 20 (2000) 197–207.