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Kernels by monochromatic paths in digraphs with covering number 2[☆]

Hortensia Galeana-Sánchez [a], Mika Olsen [b,*]

[a] Instituto de Matemáticas, UNAM, Circuito Exterior, Ciudad Universitaria, 04510 México DF, Mexico

[b] Departamento de Matemáticas Aplicadas y Sistemas, UAM-Cuajimalpa, Calle Artificios 40 6^0 piso, Álvaro Obregón, CP 01120, México DF, Mexico

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ABSTRACT

We call the digraph D an k-colored digraph if the arcs of D are colored with k colors. A subdigraph H of D is called monochromatic if all of its arcs are colored alike. A set $N \subseteq V(D)$ is said to be a kernel by monochromatic paths if it satisfies the following two conditions: (i) for every pair of different vertices $u, v \in N$, there is no monochromatic directed path between them, and (ii) for every vertex $x \in (V(D) \setminus N)$, there is a vertex $y \in N$ such that there is an xy-monochromatic directed path. In this paper, we prove that if D is an k-colored digraph that can be partitioned into two vertex-disjoint transitive tournaments such that every directed cycle of length 3, 4 or 5 is monochromatic, then D has a kernel by monochromatic paths. This result gives a positive answer (for this family of digraphs) of the following question, which has motivated many results in monochromatic kernel theory: *Is there a natural number l such that if a digraph D is k-colored so that every directed cycle of length at most l is monochromatic, then D has a kernel by monochromatic paths?*

1. Introduction

Let D be a digraph. We denote by $V(D)$ and $A(D)$ the sets of vertices and the set of arcs of D, respectively. Let $v \in V(D)$. We denote by $N^+(v)$ and $N^-(v)$ the out- and in-neighborhood of v in D, respectively. We define $\delta^+(w) = |N^+(w)|$ and $\delta^-(w) = |N^-(w)|$. For $S \subseteq V(D)$, we denote by $D[S]$ the subdigraph of D induced by the vertex set S. For two disjoint subsets U, V of $V(D)$, we denote by $(U, V) = \{uv \in A(D) : u \in U, v \in V\}$ and $[U, V] = (U, V) \cup (V, U)$. An UV-arc is an arc from (U, V) if $U = \{u\}$ (resp. $V = \{v\}$), we denote the UV-arc by uV-arc (resp. Uv-arc). We call the digraph D an *k-colored* digraph if the arcs of D are colored with k colors. The digraph D will be an k-colored digraph and all the paths, cycles and walks considered in this paper will be directed paths, cycles or walks. If $W = (x_0, x_1, \ldots, x_n)$ is a walk, the *length* of W is n. The length of a walk W is denoted by $l(W)$. The path $(u_0, u_1, \ldots, u_n)$ will be called an UV-path whenever $u_0 \in U$ and $u_n \in V$. A *tournament* is a digraph T such that there is exactly one arc between any two vertices of T. An acyclic tournament is called a *transitive tournament*. A vertex $v \in V(T)$ is called a *sink* if $N^-(v) = V(D) \setminus v$. A subdigraph H of a k-colored digraph D is called *monochromatic* if all of its arcs are colored alike. Let $N \subseteq V(D)$. Then, N is said to be *m-independent* if there is no monochromatic directed path between any pair of vertices of the set N, N is a *m-absorbent* (or m-dominant) if for every vertex $x \in (V(D) \setminus N)$ there is a vertex $y \in N$ such that there is an xy-monochromatic directed path and finally, N is a *m-kernel* (kernel by monochromatic paths) if it satisfies the following two conditions: (i) N is m-independent and (ii) N is m-absorbent. For general concepts, we refer the reader to [1,2,7].

The topic of domination in graphs has been widely studied by many authors. A very complete study of this topic is presented in [19,20]. A special class of domination is the domination in digraphs, and it is defined as follows. In a digraph D,

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* Corresponding author.
E-mail addresses: olsen@correo.cua.uam.mx, olsen@correo.cua.mx (M. Olsen).

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a set of vertices $S \subseteq V(D)$ dominates whenever for every $w \in (V(D) \setminus S)$ there exists a wS-arc in D. Dominating independent sets in digraphs (kernels in digraphs) have found many applications in different topics of mathematics (see for instance [21,22,8,9,26]) and they have been studied by several authors; interesting surveys of kernels in digraphs can be found in [6,9]. The concepts of m-domination, m-independence and m-kernel in edge-colored digraphs are generalization of those of domination, independence and kernel in digraphs. The study of the existence of m-kernels in edge-colored digraphs starts with the theorem of Sands, Sauer and Woodrow, proved in [25], which asserts that every two-colored digraph possesses an m-kernel. In the same work, the authors proposed the following question: let D be an k-colored tournament such that every directed cycle of length 3 is quasi-monochromatic (a subdigraph H of an k-colored digraph D is said to be *quasi-monochromatic* if, with at most one exception, all of its arcs are colored alike) must D have a m-kernel? Minggan [24] proved that if D is an k-colored tournament such that every directed cycle of length 3 and every transitive tournament of order 3 is quasi-monochromatic, then D has a m-kernel. He also proved that this result is best possible for $m \geq 5$. In [15], it was proved that the result is best possible for $m \geq 4$. The question for $m = 3$ is still open: *Does every* 3-*colored tournament such that every directed cycle of length* 3 *is quasi-monochromatic have a m-kernel*? Sufficient conditions for the existence of m-kernels in edge-colored digraphs have been obtained mainly in tournaments and generalized tournaments, and ask for the monochromaticity or quasi-monochromaticity of small digraphs (due to the difficulty of the problem) in several papers (see [10,11,15,16,18,24]).Other interesting results can be found in [27,28]. Another question which has motivated many results in m-kernel theory is the following (proposed in the abstract): *Given a digraph D is there an integer k such that if every directed cycle of length at most k is monochromatic (resp. quasi-monochromatic), then D has a m-kernel*? In [11] (resp. in [16]) it was proved that if D is an k-colored tournament (resp. bipartite tournament) such that every directed cycle of length 3 (resp. every directed cycle of length 4) is monochromatic, then D has a m-kernel. Later the following generalization of both results was proved in [17]: if D is an k-colored k-partite tournament, such that every directed cycle of length 3 and every directed cycle of length 4 is monochromatic, then D has a m-kernel. In [18] were considered quasi-monochromatic cycles, the authors proved that if D is an k-colored tournament such that for some k every directed cycle of length k is quasi-monochromatic and every directed cycle of length less than k is not polychromatic (a subdigraph H of D is called *polychromatic* whenever it is colored with at least three colors), then D has a m-kernel. In [13] this result was extended for nearly complete digraphs. The *covering number* of a digraph D is the minimum number of transitive tournaments of D that partition $V(D)$. Digraphs with a small covering number are a nice class of nearly tournament digraphs. The existence of kernels in digraphs with a covering number at most 3 has been studied by several authors, in particular by Berge [3], Maffray [23] and others [4,5,12,14].

In this paper, we study the existence of m-kernel in edge-colored digraphs with covering number 2, asking for the monochromaticity of small directed cycles. We prove that if D is an k-colored digraph with covering number 2 such that every directed cycle of length 3, 4 or 5 is monochromatic, then D has a m-kernel.

2. Structural properties

We consider the family $\mathfrak{D}$ of digraphs D with covering number 2. Since D has covering number 2, there exists a non-trivial partition of $V(D)$ into two sets U, V such that $D[U], D[V]$ are transitive tournaments. Throughout this paper, the non-trivial partition of the vertex set into U, V is such that $D[U], D[V]$ are transitive tournaments. Let T be a transitive tournament of order n. Throughout this paper, $(v_n, v_{n-1}, \ldots, v_1)$ will denote the Hamiltonian path in T. Thus for any $1 \leq i \leq n$, the vertex v_i is the sink of $T \setminus \{v_1, v_2, \ldots, v_{i-1}\}$, in particular, v_1 is the sink of T. When $P = (u_0, u_1, \ldots, u_k)$ is a path, we will denote by (u_i, P, u_j), for $0 \leq i < j \leq k$, the u_i, u_j-path contained in P. Let u_iu_{i+1} and u_ju_{j+1} be two distinct arcs on P. We say that the arc u_iu_{i+1} *precedes* (resp. *follows*) the arc u_ju_{j+1} on the path P, if $i < j$ (resp. $j < i$).

Throughout this paper, the vertex z will be **fixed and arbitrary**.

First, we prove some structural properties of the wz-paths of minimum length with $w \in \{u_1, v_1\}$ in digraphs of the family $\mathfrak{D}$. Next, we extend these properties for wz-paths of minimum length with $w \in \{u_1, v_1\}$ in k-colored digraphs of the family $\mathfrak{D}$ with every directed cycle of length 3, 4 or 5 monochromatic.

Let $u_mv_n, v_ku_l \in [U, V]$. We say that u_mv_n, v_ku_l are *crossing arcs* if $u_m, u_l \in V(D)$, $v_n, v_k \in V(D)$, and $m \leq l, k \leq n$, except when $n = k$ and $m = l$ (see Fig. 1). Let $u_mv_n \in (U, V)$. If $xy, u_mv_n \in [U, V]$ are crossing arcs, then clearly $xy \in (V, U)$.

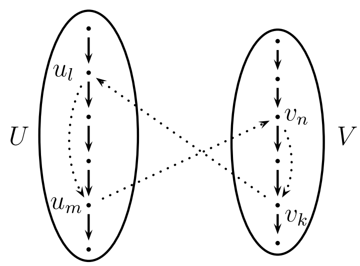

Fig. 1. u_mv_n and v_ku_l are crossing arcs.

Lemma 1. *Let D be a digraph of the family $\mathfrak{D}$ such that the sinks u_1, v_1 of $D[U]$ and $D[V]$ respectively has a nonempty out-neighborhood. Let P be a wz-path of minimum length, with $w \in \{u_1, v_1\}$. Then, any $a \in [U, V] \cap A(P)$ has at most one preceding crossing arc on P and at most one following crossing arc on P.*

Proof. Let D be a digraph that satisfies the hypothesis of this lemma and let P be a wz-path of minimum length starting at the vertex u_1 or v_1 and $|[U, V] \cap A(P)| \geq 2$.

Suppose, for a contradiction, that there is an arc of $[U, V] \cap A(P)$ with at least two following (*preceding*) crossing arcs on P. By symmetry, we may assume that $u_iv_j \in (U, V) \cap A(P)$ is such that u_iv_j has at least two following (*preceding*) crossing arcs in (V, U). Let $v_ku_l \in (V, U)$ be the first following crossing arc on P and let $v_su_t \in (V, U)$ be the last following crossing arc on P (*resp. let $v_eu_f \in (V, U)$ be the first preceding crossing arc on P*). Since u_iv_j and v_su_t are crossing arcs, $j \geq k$, s and $l, t \geq i$. Do also note that P is a path and therefore $l \neq t$ and $k \neq s$ (see Fig. 2).

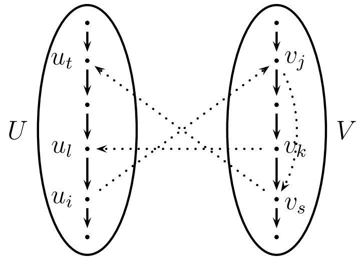

Fig. 2. u_iv_j and v_su_t are crossing arcs.

The cycle $(u_i, v_j, v_s, u_t, u_i)$ (*resp. the cycle* $(u_i, v_j, v_e, u_f, u_i)$) has length at most 4 (see Fig. 2), and since P is a path, v_jv_s is not on P, then $P' = (w, P, v_j) \cup (v_j, v_s) \cup (v_s, P, z)$ (*resp.* $P' = (w, P, u_f) \cup (u_f, u_i) \cup (u_i, P, z)$) is a wz-directed path such that $l(P') < l(P)$, which contradicts the way we chose P. So $u_iv_j \in (U, V)$ has at most one following (*preceding*) crossing arc; by symmetry any arc of $[U, V]$ has at most one following (*preceding*) crossing arc and we are done. □

Remark 1. Let D be a digraph of the family $\mathfrak{D}$ such that $\delta^+(u_1), \delta^+(v_1) > 0$. Let P be a wz-path of minimum length with $w \in \{u_1, v_1\}$. Then, xy is the first arc of $[U, V] \cap A(P)$ if and only if $x = w$.

Lemma 2. *Let D be a digraph of the family $\mathfrak{D}$ such that $\delta^+(u_1), \delta^+(v_1) > 0$. Let P be a directed wz-path of minimum length, with $w \in \{u_1, v_1\}$. Then, any two consecutive $[U, V]$-arcs on P are crossing arcs.*

Proof. Let D be a digraph that satisfies the hypothesis of this lemma and let P be a path of minimum length, starting at the vertex u_1 or v_1, with $|[U, V] \cap A(P)| \geq 2$.

Suppose, for a contradiction, that there are two consecutive $[U, V]$-arcs on P such that they are not crossing arcs. By symmetry, we may assume that $u_iv_j \in (U, V)$, $v_ku_l \in (V, U)$ is the first pair of consecutive $[U, V]$-arcs that does not form a crossing pair, then $1 \leq l < i$ and by Remark 1, the arc u_iv_j must have a preceding VU-arc, and this arc must be a crossing arc by the way we chose the arc u_iv_j. Let $v_gu_h \in (V, U)$ be the preceding crossing arc of u_iv_j. Since $D[U]$ is transitive, then $P' = (w, P, u_h) \cup (u_h, u_l) \cup (u_l, P, z)$ is a wz-path such that $l(P') < l(P)$ (see Fig. 3), which is a contradiction. □

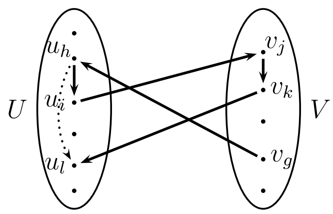

Fig. 3. How we find a path P' such that $l(P') < l(P)$.

Proposition 1. *Let D be a digraph of the family $\mathfrak{D}$ such that $\delta^+(u_1), \delta^+(v_1) > 0$. Let P be a wz-path of minimum length with $|[U, V] \cap A(P)| \geq 2$. Then, the first and the last $[U, V]$-arc on P have exactly one crossing arc and any $[U, V]$-arc except the first and the last one, has exactly one preceding and exactly one following crossing $[U, V]$-arc.*

Proof. A consequence of the Lemmas 1 and 2. □

We will now extend the results Lemmas 1, 2 and Proposition 1 to k-colored digraphs of the family $\mathfrak{D}$, with every directed cycle of length 3, 4 or 5 monochromatic.

Remark 2. Let D be an k-colored digraph of the family $\mathfrak{D}$ such that the cycles of length 3,4 and 5 are monochromatic. Then any two crossing arcs have the same color, and so if $m < l$ and $k < n$, then $(u_l, u_m, v_n, v_k, u_l)$ is a 4-cycle, if $m < l$ and $n = k$, then (u_m, v_n, u_l, u_m) is a 3-cycle and if $m = l$ and $k < n$, then (u_m, v_n, v_k, u_m) is a 3-cycle (see Fig. 1).

Moreover for any integers i or j, $m < i < l$ and $n < j < k$ (if they exist), the arcs $u_l u_i$, $u_i u_m$, $v_n v_j$, $v_j v_k$ have the same color as the crossing arcs $u_m v_n$, $v_k u_l$ (see Fig. 1).

Lemma 3. *Let D be an k-colored digraph of the family $\mathfrak{D}$ such that the cycles of length 3, 4 and 5 are monochromatic. If $\delta^+(u_1), \delta^+(v_1) > 0$, then $u_1 v_j$ and $v_1 u_i$ has the same color for any $u_i \in N^+(v_1)$ and $v_j \in N^+(u_1)$.*

Proof. Let i, j be two integers such that $u_i \in N^+(v_1)$ and $v_j \in N^+(u_1)$. Then, the arcs $u_1 v_j$, $v_1 u_i$ are crossing arcs and by Remark 2, they have the same color. □

Corollary 1. *Let D be an k-colored digraph of the family $\mathfrak{D}$ such that the cycles of length 3, 4 and 5 are monochromatic. If $\delta^+(u_1), \delta^+(v_1) > 0$, then all the monochromatic paths from u_1 or from v_1 have the same color.*

By Lemma 3, we may assume that every arc $u_1 v_j$, $v_1 u_i$ has color 1, and by Corollary 1, any monochromatic $u_1 x$-path, $v_1 x$-path has color 1, for all $x \in V(D)$.

Lemma 4. *Let D be an k-colored digraph of the family $\mathfrak{D}$ such that the cycles of length 3 and 4 are monochromatic and such that $\delta^+(u_1), \delta^+(v_1) > 0$. Let P be a monochromatic wz-path of minimum length, with $w \in \{u_1, v_1\}$, and let $a \in [U, V] \cap A(P)$. Then the arc a has at most one preceding crossing $[U, V]$-arc on P and at most one following crossing $[U, V]$-arc on P.*

Proof. Let D be a digraph that satisfies the hypothesis of the Lemma 4 and let P be a monochromatic wz-path of minimum length starting at the vertex u_1 or v_1 and $|[U, V] \cap A(P)| \geq 2$.

Suppose, for a contradiction, that $a \in [U, V] \cap A(P)$ such that a has at least two following crossing arcs on P. By symmetry, we may assume that $u_i v_j \in (U, V)$ is the first $[U, V]$-arc of P such that $u_i v_j$ has at least two following crossing arcs on P. Proceed as in the proof of Lemma 1. Since the cycle $(u_i, v_j, v_s, u_t, u_i)$ has length at most 4, it is monochromatic. Moreover, the arc $u_i v_j$ has color 1, so the cycle and the path P' are both monochromatic of color 1 (see Fig. 2) and P' is a monochromatic wz-path such that $l(P') < l(P)$. So any $[U, V]$-arc on P has at most one following crossing $[U, V]$-arc on P. Analogously, any $[U, V]$-arc on P has at most one preceding crossing $[U, V]$-arc on P. □

Remark 3. Let D be an k-colored digraph of the family $\mathfrak{D}$ such that the cycles of length 3, 4 and 5 are monochromatic. Let P be a monochromatic wz-path of minimum length with $w \in \{u_1, v_1\}$. Then xy is the first $[U, V]$-arc on P if and only if $x = w$.

Lemma 5. *Let D be an k-colored digraph of the family $\mathfrak{D}$ such that the cycles of length 3, 4 and 5 are monochromatic and such that $\delta^+(u_1), \delta^+(v_1) > 0$. Let P be a monochromatic wz-path of minimum length, with $w \in \{u_1, v_1\}$. Then any two consecutive $[U, V]$-arcs on P are crossing arcs.*

Proof. Let D be a digraph that satisfies the hypothesis of this lemma and let P be a monochromatic path of minimum length starting at the vertex u_1 or v_1, with $|[U, V] \cap A(P)| \geq 2$.

Suppose, for a contradiction, that there are two consecutive $[U, V]$-arcs on P that are not crossing arcs. By symmetry, we may assume that $u_i v_j \in (U, V)$ is the first $[U, V]$-arc such that the following $[U, V]$-arc $v_k u_l$ is not a crossing arc of $u_i v_j$.

Claim 1. *If $v_1 u_x \in A(P)$ is the first $[U, V]$-arc on P, then $x < l$.*

Let $x \geq l$. Suppose, for a contradiction, that $v_1 u_x$ is the first $[U, V]$-arc on P, then $(v_1, u_x, u_l, u_1, v_y, v_1)$ is a monochromatic cycle of color 1 and of length at most 5 (where v_y is any vertex in $N^+(u_1)$). Thus $P' = (w = v_1, u_x, u_l) \cup (u_l, P, z)$ is a directed monochromatic wz-path of color 1. Since $|[U, V] \cap A(P)| \geq 2$, P' is such that $l(P') < l(P)$, and the Claim 1 is valid. □

Claim 2. *P has no crossing arcs $u_a v_b$, $v_c u_d$ preceding the arc $u_i v_j$ with $a < l < d$.*

Let $a < l < d$. For a contradiction, suppose that $u_a v_b$, $v_c u_d$ is a pair of crossing arcs, both preceding the arc $u_i v_j$ on the path P. The length of the cycle $C = (u_a, v_b, v_c, u_d, u_l, u_a)$ is at most 5. The path P is monochromatic of color 1, and so is the cycle C. If $u_a v_b$, $v_c u_d$ are consecutive crossing arcs, then $P' = (w, P, u_d) \cup (u_d, u_l) \cup (u_l, P, z)$ is a directed monochromatic wz-path. The arcs $u_a v_b$, $v_c u_d$ are preceding to the arc $u_i v_j$; thus, P' is such that $l(P') < l(P)$. If $v_c u_d$, $u_a v_b$ are consecutive crossing arcs, then $P' = (w, P, u_d) \cup (u_d, u_l) \cup (u_l, P, z)$ is a directed monochromatic wz-path. The arcs $u_a v_b$, $v_c u_d$ are preceding to the arc $u_i v_j$; thus, P' is such that $l(P') < l(P)$. So Claim 2 is valid. □

The arcs $u_i v_j$ and $v_k u_l$ are not crossing arcs, then $i > l > 0$ and by Remark 3, the arc $u_i v_j$ is not the first $[U, V]$-arc on P. Let $v_g u_h \in (V, U)$ be the preceding crossing arc of $u_i v_j$. Then, $h \geq i > l$ and $g < j$.

Note that $h \geq i > l$. By Claim 1 and Remark 3, the arc $v_g u_h$ is not the first $[U, V]$-arc on P. Let $v_e u_f \in (V, U)$ be the first arc on P such that $f > l$, such arc does exist (for instance $v_g u_n$). By Claim 1 and Remark 3, the arc $v_e u_f$ is not the first $[U, V]$-arc on P. Then, the arc $v_e u_f$ must have a preceding $[U, V]$-arc, and this arc must be a crossing arc by the way we chose the arc $u_i v_j$. Let $u_c v_d \in (U, V)$ be the preceding crossing arc of $v_e u_f$. By Claim 2 and the fact that $f > l$, we have that $c \geq l$; moreover, since P is a path $c > l$, then Claim 1 and Remark 3 imply that the arc $u_c v_d$ is not the first $[U, V]$-arc on P. So the arc $v_e u_f$

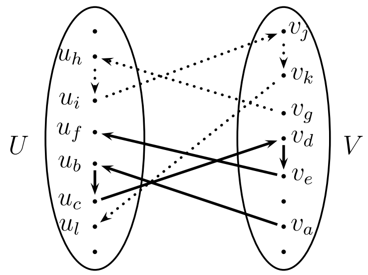

Fig. 4. Any two consecutive $[U, V]$-arcs on P are crossing arcs.

must have a preceding $[U, V]$-arc, and this arc must be a crossing arc by the way we chose the arc $u_i v_j$. Let $v_a u_b \in (V, U)$ be the preceding crossing arc of $u_c v_d$ (see Fig. 4).

By the choice of the arc $u_i v_j$, it follows that $v_a u_b$ and $u_c v_d$ are crossing arcs, and $b > c > l$ which contradicts the choice of the arc $v_e u_f$. So any two consecutive $[U, V]$-arcs on P does form a crossing pair of arcs. □

Corollary 2. *Let D be an k-colored digraph of the family $\mathfrak{D}$ such that the cycles of length 3, 4 and 5 are monochromatic and such that $\delta^+(u_1), \delta^+(v_1) > 0$. Let P be a monochromatic wz-path of minimum length with $|[U, V] \cap A(P)| \geq 2$. Then, the first and the last $[U, V]$-arc on P have exactly one crossing arc and any $[U, V]$-arc, except the first and the last one has exactly one preceding and exactly one following crossing $[U, V]$-arc. Moreover, two crossing arcs on P must be consecutive $[U, V]$-arcs on P.*

Proof. A consequence of the Lemmas 4 and 5. □

The following theorem collects the results of Lemmas 4, 5 and Corollary 2. Moreover, it describes the structure of a monochromatic path of minimum length, as shown in Fig. 5.

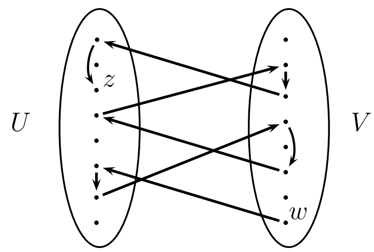

Fig. 5. The structure of a monochromatic directed wz-path of minimum length.

We denote by P_U the digraph on the vertex set $V(P) \cap U$, and $A(P_U) = A(P) \cap A(D[U])$.

Theorem 1. *Let D be an k-colored digraph of the family $\mathfrak{D}$ such that the cycles of length 3, 4 and 5 are monochromatic and such that $\delta^+(u_1), \delta^+(v_1) > 0$. Let P be a monochromatic wz-path of minimum length, with $w \in \{u_1, v_1\}$. Then,*

(i) *if $u_a v_b \in (U, V)$ and $u_e v_f \in (U, V)$ are arcs on P and $u_e v_f$ follows $u_a v_b$, $a < e$ and $b < f$;*
(ii) *an induced path of P_U (resp. P_V) has length at most one.*

Proof. Let D be a digraph that satisfies the hypothesis of this lemma and let P be a monochromatic path of minimum length starting at the vertex u_1 or v_1.

(i) In order to prove the item (i) we take $u_a v_b$ and $u_e v_f$ the preceding and the following crossing arc respectively of $v_c u_d$ on the path P, by the definition of crossing arcs $a, e < d$ and $c < b, f$. Suppose, for a contradiction, that $a > e$, then $u_a v_b$ is not the first $[U, V]$-arc on P, by Remark 3. By Lemma 5, $u_a v_b$ has a preceding crossing arc, say h, then the arc $u_e v_f$ would have two preceding crossing arcs, namely $v_c u_d$ and h and by Lemma 4, we have a contradiction, so $a < e$. Analogously $b < f$.
(ii) In order to prove that an induced path of P_U has length at most 1, we take two consecutive crossing arcs on the path P, say $u_a v_b$, $v_c u_d$, and prove that $v_b v_c \in A(P)$. The length of the cycle $C = (u_a, v_b, v_c, u_d, u_a)$ is at most 4 and the path P is monochromatic of color 1, and so is the cycle C. Then $P' = (w, P, v_b) \cup (v_b, v_c) \cup (v_c, P, z)$ is a directed monochromatic wz-path. If $v_b v_c \notin A(P)$, then P' would be a monochromatic zw-path such that $l(P') < l(P)$, so $v_b v_c \in A(P)$. □

3. *m*-kernel

Let D be an k-colored digraph. A subset S of $V(D)$ is a *m-semi-kernel* of D if it satisfies the following two conditions:

(a) S is m-independent, and
(b) for every vertex $z \notin S$ for which there exists a Sz-monochromatic directed path, there also exists a zS-monochromatic directed path.

A kernel of a digraph D is also a semi-kernel of D, but the converse is not true.

We prove that an k-colored digraph D of the family $\mathfrak{D}$ with any cycle of length 3, 4 and 5 monochromatic has a m-semi-kernel of only one vertex. This fact will lead us to the main theorem.

The main idea in the proof of Proposition 2 is the following.

Let x be any vertex, say u_s, on a monochromatic wz-path P of minimum length, with $w \in \{u_1, v_1\}$. We prove that if P has at least two $[U, V]$-arcs, then there is a pair of crossing arcs on P, say $u_i v_j$ and $v_k u_l$, such that $i < l$ and $i \leq s \leq l$.

Proposition 2. *Let D be an k-colored digraph of the family $\mathfrak{D}$ such that the cycles of length 3, 4 and 5 are monochromatic. Then, u_1 (resp. v_1) is a m-semi-kernel of one vertex of D.*

Moreover if there is a monochromatic wz-path, with $w \in \{u_1, v_1\}$, of color 1, then there is a monochromatic zw-path of color 1.

Proof. If $\delta^+(u_1) = 0$ ($\delta^+(v_1) = 0$), then u_1 (*resp.* v_1) is a m-semi-kernel. Let $\delta^+(u_1), \delta^+(v_1) > 0$, and let k, l be maximum integers such that $u_k \in N^+(v_1)$ and $v_l \in N^+(u_1)$. By Corollary 1, we may assume that any monochromatic path from v_1 or u_1 has color 1.

We prove that if there is a monochromatic wz-path, with $w \in \{u_1, v_1\}$, then there is a monochromatic zw-path of color 1. Since u_1 is the sink of $D[U]$ (*resp.* v_1 *is the sink of* $D[V]$), there is an uu_1 arc in D, for any $u \in U \setminus u_1$ (*resp. there is an* vv_1 *arc in* D *for any* $v \in U \setminus v_1$), but this arc is not necessarily of color 1.

Proceeding by contradiction, we take a monochromatic wz-path P of minimum length (thus P is colored 1) with z as the first vertex on P such that there is no monochromatic zw-path colored 1. By symmetry, we may assume that $z = u_s$ for some integer $1 \leq s \leq n$. Let $v_k u_l \in (V, U)$ be the first arc on P such that $l \geq s$. Such arc does exist because $D[U]$ is a transitive tournament and thus for each $u_r \in U \cap V(P)$, the preceding vertex on P is a vertex of the set $\{u_{r+1}, u_{r+2}, \ldots, u_n\} \cup V$. If $v_k u_l$ is the first $[U, V]$-arc on P, then $k = 1$ and for any $v \in N^+(u_1)$ the cycle $(v_1 = v_k, u_l, u_s, u_1, v, v_1)$ has length at most 5 and is monochromatic of color 1. Then $P' = (u_s, u_1)$ (resp. $P'' = (u_s, u_1, v, v_1)$) is a monochromatic $u_s u_1$-path (resp. $u_s v_1$-path) of color 1 and we are done.

Therefore, $v_k u_l$ is not the first $[U, V]$-arc on P. Let $u_i v_j \in (U, V)$ be the preceding crossing arc on P; this arc exists by Lemma 5. Since P is a path, $i \neq l$. If $i > s$, then $u_i v_j$ is not the first $[U, V]$-arc on P and by Lemma 5, $u_i v_j$ has a preceding crossing arc $v_g u_h \in (V, U)$. By (ii) of Theorem 1, $i < h < l$, which contradicts the choice of the arc $v_k u_l$. Then, $i < s$. The cycle $(v_k, u_l, u_s, u_i, v_j, v_k)$ has length at most 5 and it is monochromatic of color 1 (see Fig. 6). By the choice of the vertex u_s, there is a monochromatic $u_i w$-path P' colored 1. Then, $(u_s, u_i) \cup P'$ is a monochromatic $u_s w$-path colored 1, and we are done.

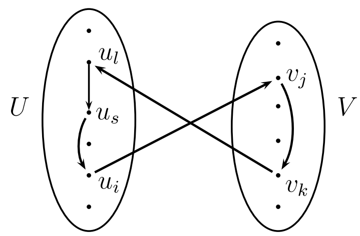

Fig. 6. The 5-cycle $(v_k, u_l, u_s, u_i, v_j, v_k)$ is monochromatic of color 1.

So, if there is a monochromatic wu_s-path, with $w \in \{u_1, v_1\}$, then there is a monochromatic $u_s w$-path of color 1. Analogously, if there is a monochromatic wv_s-path, with $w \in \{u_1, v_1\}$, then there is a monochromatic $v_s w$-path of color 1. Therefore, $\{u_1\}$ and $\{v_1\}$ are both semi-kernels of D. □

Theorem 2. *Let D be an k-colored digraph of the family $\mathfrak{D}$ such that the cycles of length 3, 4 and 5 are monochromatic. Then, D has a m-kernel.*

Proof. If $\delta^+(u_1) = 0$ (resp. $\delta^+(v_1) = 0$), then u_1 (resp. v_1) is a m-semi-kernel; else Proposition 2 implies that u_1 or v_1 is a m-semi-kernel of D. We may assume that v_1 is a m-semi-kernel of D. Suppose that v_1 is not a m-kernel of D. Let U' be the subset of the vertices of $V(D)$ such that there is no monochromatic $U'v_1$-directed path. As v_1 is a semi-kernel of D, we have that there are no monochromatic directed path between v_1 and a vertex $x \in U'$. Since $D[V]$ is a transitive tournament, $v_j v_1 \in A(D)$ for every $1 < j \leq m$; therefore, $U' \subset U$. As v_1 is not a m-kernel of D, then $U' \neq \emptyset$ and $D[U]$ is a transitive tournament, then $D[U']$ has a sink. Let u_p be the sink of $D[U']$. Then $\{v_1, u_p\}$ is a kernel by monochromatic paths of D. □

4. Final remarks

In this section, we show three digraphs from the family $\mathfrak{D}$. The first one (Example 1) is a digraph colored with a large number of colors. Next we show two digraphs without m-kernel; the first one (Example 2) has 4- and 5-cycles that are not monochromatic and the second one (Example 3) has 3- and 5-cycles that are not monochromatic. The Example 2 shows that the condition of monochromatic 3-cycles is not sufficient, and Example 3 shows that monochromatic 4-cycles is not sufficient.

Example 1. We define the digraph D as follows.

Let $U = \{u_1, u_2, \ldots, u_k\}$, $V = \{v_1, v_2, \ldots, v_l\}$ be a partition of the vertex set $V(D)$ such that $D[U]$, $D[V]$ are transitive tournaments. Let

$$A(D) = \{u_iu_j : j < i\} \cup \{v_iv_j : j < i\} \cup \{v_1u_2, u_1v_2\} \cup A'$$
$$A' \subset \{u_iv_j : j \leq i\} \cup \{v_iu_j : j < i\} \setminus \{v_2u_1, u_2v_1\}.$$

The only cycles of D are the cycles in $D[u_1, u_2, v_1, v_2]$. Since there are no other cycles, we can color each arc outside $D[u_1, u_2, v_1, v_2]$ with a different color and still have an arc coloring of D, with all cycles of length 3, 4 and 5 monochromatic.

If D is a tournament, then D has $\binom{k+l}{2}$ arcs. There are 6-arcs in $D[u_1, u_2, v_1, v_2]$, so the maximum number of colors of D is

$$\binom{k+l}{2} - 5.$$

So, we have k-colored digraphs in the family $\mathfrak{D}$ with any cycle of length 3, 4 and 5 monochromatic, such that the D is not a tournament nor a nearly complete digraphs, and $m = A(D) - 6$.

Example 2. Let D be the digraph in Fig. 7(a). Note that the 4-cycle $(u_1, v_3, u_4, u_2, u_1)$ is not monochromatic. We show that D has no m-kernel. First observe that the only vertex that absorbs the vertex v_2 is v_1. If D has a kernel K, then v_1 or v_2 are vertices of K, but not both. Suppose that $v_1 \in K$. Since (v_1, u_3, u_1) is a monochromatic path, $u_1 \notin K$. The only vertices that absorbs the vertex u_1 are the vertices u_4 and v_3, but v_3 is not independent to v_1 and u_4 does not absorb the vertex u_2; then $v_1 \notin K$ and $v_2 \in K$. In this case, v_1 is not absorbed by v_2. The vertices of D that absorb the vertex v_1 are u_1 and u_3, but u_1 is not independent to v_2 and u_3 does not absorb the vertex u_2, so $v_2 \notin K$. Therefore, D has no kernel.

Example 3. Let D be the digraph in Fig. 7(b). Note that the 3-cycle (u_2, v_3, u_3, u_2) and the 5-cycle $(u_2, v_3, u_3, v_4, u_4, u_2)$ are not monochromatic. We show that D has no m-kernel. First observe that there is no vertex $x \in V(D)$ such that x absorbs every vertex from $V(D) \setminus x$. Thus, if D has a kernel K, then $|K| \geq 2$ and $K \cap \{u_1, u_2, u_3, u_4\} \neq \emptyset$ and $K \cap \{v_1, v_2, v_3, v_4\} \neq \emptyset$. Any vertex of the set $\{v_1, v_2, v_3\}$ absorbs all the vertices of D except the vertices u_3, u_4. If $\{v_1, v_2, v_3\} \cap K \neq \emptyset$, then $u_3 \notin K$, because there is a monochromatic path from any of the vertices v_1, v_2, v_3 to the vertex u_3. Thus, $u_4 \in K$, but the vertex u_4 does not absorb the vertex u_3. Therefore, $\{v_1, v_2, v_3\} \cap K = \emptyset$ and$v_4 \in K$. The vertex v_4 absorbs all the vertices except the vertex u_4, but v_4u_4 is a directed monochromatic path. So, the digraph D has no m-kernel.

In Fig. 7 there are two digraphs, 3- and 4-colored respectively, with the colors.

In Fig. 7(a) the 3-cycles are all monochromatic, but there are 4-cycles and 5-cycles, which are not monochromatic (for instance $(u_1, v_3, u_4, u_2, u_1)$). In Fig. 7(b) the 4-cycles are all monochromatic, but there are 3-cycles and 5-cycles which are not monochromatic. In both cases, the digraph has no m-kernel. These digraphs shows that monochromatic 3-cycles are not sufficient and that monochromatic 4-cycles are not sufficient.

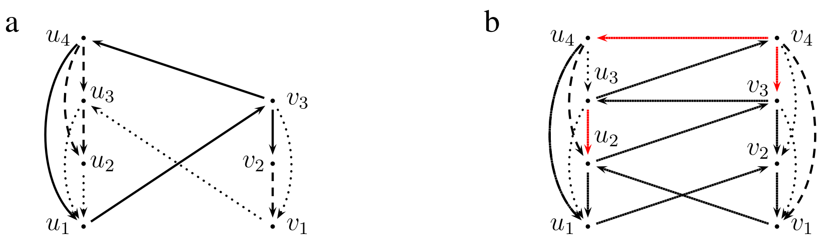

Fig. 7. Digraphs without a m-kernel.

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