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A conjecture of Neumann-Lara on infinite families of r-dichromatic circulant tournaments

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ABSTRACT

In this paper, we exhibit infinite families of vertex critical r-dichromatic circulant tournaments for all $r \geq 3$. The existence of these infinite families was conjectured by Neumann-Lara [V. Neumann-Lara, Note on vertex critical 4-dichromatic circulant tournaments, Discrete Math. 170 (1997) 289–291], who later proved it for all $r \geq 3$ and $r \neq 7$. Using different methods, we provide new constructions of such infinite families for all $r \geq 3$, which covers the case $r = 7$ and thus settles the conjecture.

1. Introduction

Let D be a digraph and $V(D)$, $A(D)$ the sets of vertices and arcs of D respectively. The digraph D is *acyclic* if it contains no directed cycles. A subset of $V(D)$ which induces an acyclic subdigraph of D will be called *acyclic*. An acyclic tournament of order n is called a transitive tournament and is denoted by TT_n. We say that a tournament T is TT_n-free if no TT_n is isomorphic to any subdigraph of T.

The dichromatic number of a digraph D, denoted $dc(D)$, was introduced in [5] as the minimum number of colors needed to color the vertices of D so that each chromatic class induces an acyclic subdigraph of D. A digraph is called *r-dichromatic* if $dc(D) = r$, and *vertex critical r-dichromatic* if $dc(D) = r$ and $dc(D \setminus v) < r$ for all $v \in V(D)$.

Let $\mathbb{Z}_{2n+1}$ be the set of integers $\mod(2n+1)$ and J a subset of $\mathbb{Z}_{2n+1}\setminus\{0\}$ such that $|\{j, -j\} \cap J| = 1$ for every $j \in \mathbb{Z}_{2n+1}\setminus\{0\}$. The *circulant tournament* $\overrightarrow{C}_{2n+1}(J)$ is defined as follows: $V(\vec{C}_{2n+1}(J)) = \mathbb{Z}_{2n+1}$ and $A(\vec{C}_{2n+1}(J)) = \{(i, j) : i, j \in \mathbb{Z}_{2n+1}, j - i \in J\}$. Recall that the circulant tournaments are vertex transitive.

The vertex critical r-dichromatic tournaments have been studied in several papers [7,9,10]. In [7] Neumann-Lara conjectured that there exists an infinite family of r-dichromatic vertex critical circulant tournaments for all $r \geq 3$. He proved this conjecture for all $r \geq 3$ and $r \neq 7$ [9], using composition of tournaments.

In this paper, we explicitly exhibit infinite families of vertex critical r-dichromatic circulant tournaments for each $r \geq 3$. The conjecture is then completely settled since our construction includes the case $r = 7$.

It is important to note that we do not use compositions of tournaments to construct these infinite families. We generalize the infinite families of 3 and 4 dichromatic vertex critical circulant tournaments presented in [7,10]. The infinite families constructed in this paper for $r \geq 5$, are not isomorphic to the ones in [9]. Moreover, our infinite families include the special tournaments ST_7, ST_{13}, the largest circulant tournaments free of transitive subtournaments of order 4 [3] and 5 [11], and the special tournament ST_{31} [12]. Moreover ST_7 is the 3-dichromatic circulant tournament of minimum order [6].

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In the second section we construct specific tournaments. Each tournament is encoded by a suitable matrix which naturally reflects its adjacency structure.

In the third section, we prove that, for each $n \geq 2$, there exists an infinite family of $(n+1)$-dichromatic vertex critical circulant tournaments of order $n(p+1)+1$ with $p = (n-1)(s+1)+1$, $s \geq 0$, in which the induced graph by the outneighborhood of each vertex is isomorphic to the tournament introduced in the second section. Moreover, we prove that all elements of the family do not contain transitive subtournaments of order $p+2$, but they have a decomposition into n disjoint transitive tournaments of order $p+1$ and one isolated vertex. This fact leads immediately to an $(n+1)$-vertex coloring which is critical by vertex transitivity. In a recent work [4] this family is used to prove the existence of an infinite family of tight r-dichromatic circulant tournaments for any $r \geq 2$, which is a special case of the "For every couple of integers $(r; s)$ such that $r \geq s \geq 2$ there is an infinite set of circulant tournaments T such that $dc(T) = r$ and $w_3 = s$ ($w_3 = s$ resp.) proposed by Neumann–Lara in [8]".

For general concepts we refer the reader to [1,2].

2. The family of $\mathbf{H}_{p,s}$-triangles

In this section, we construct, for each pair of integers $p \geq 1$ and $s \geq 0$ the tournament denoted by $\mathbf{H}_{p,s}$. We use these tournaments to construct for each $n \geq 2$, $s \geq 0$, and $p = (n-1)(s+1)+1$, the circulant tournament $\vec{C}_{n(p+1)+1}(J)$ with

$$J = \{1, 2, \ldots, p\} \cup \{p+2, p+3, \ldots, 2p-s\} \cup \{2p+3, \ldots, 3p-2s\} \cup \ldots \cup \{(n-1)p+n\}$$
$$= \bigcup_{k=0}^{n-1} \{kp+(k+1), \ldots, kp+p-ks\}.$$

Fig. 1 describes the in and outneighborhood of the vertex 0 in the tournament $\vec{C}_{193}(J)$, in this case $p = 23$, $n = 8$, $s = 2$. In general, we construct a figure with the same properties to describe the in and outneighborhood of the vertex 0 in the tournament $\vec{C}_{n(p+1)+1}(J)$; the outneighbors of the vertex 0 are the points $\{*, \bullet, \circ\}$ (that will be the vertex set $V(\mathbf{H}_{p,s})$) and the inneighbors of the vertex 0 are the points $\{.\}$. We prove that $\mathbf{H}_{p,s}$ is isomorphic to the tournament induces by $N^+(i)$, where i is any vertex of $\vec{C}_{n(p+1)+1}(J)$.

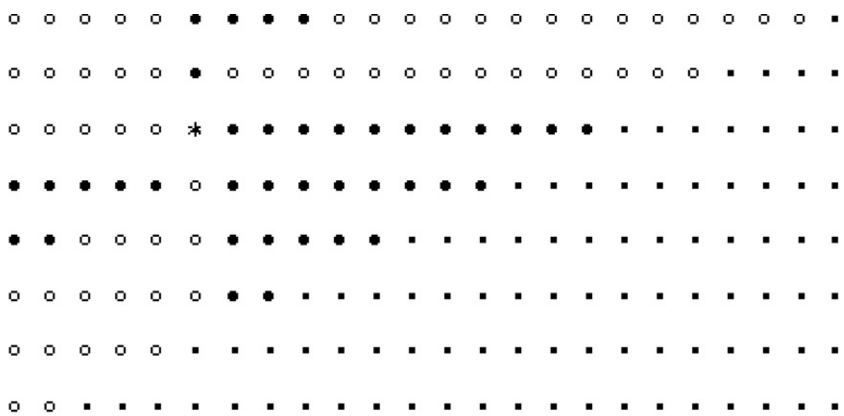

Fig. 1. The tournament $\mathbf{H}_{23,2}$. In this example $m_{ij} = m_{3,6}$.

First, we define the tournament $\mathbf{H}_{p,s}$ and prove some of its properties. Let $p := (n-1)(s+1)+r$, with $n \geq 1$ and $0 < r \leq s+1$. We define the set of vertices $V(\mathbf{H}_{p,s})$, denoted by $H_{p,s}$, as the following subset of entries m_{ij} in an $M_{n\times(p+1)}$ matrix,

$$H_{p,s} = \left\{ m_{ij} \in M_{n\times(p+1)} : (i-1)(s+1)+j \leq p \right\}.$$

Now, we define the outneighborhood of each vertex $m_{ij} \in H_{p,s}$ as the union of three disjoint sets of vertices in $H_{p,s}$, specifically $N^+(m_{ij}, \mathbf{H}_{p,s}) = A_{ij} \cup B_{ij} \cup C_{ij}$, where

$$A_{ij} = \left\{ m_{kl} \in M_{n\times(p+1)} : (k-1)(s+1)+l-j \leq (i-2)(s+1), k \geq 1, l \geq j \right\},$$
$$B_{ij} = \left\{ m_{kl} \in M_{n\times(p+1)} : (k-i)(s+1)+l-j-1 < (n-i)(s+1)-j+r, k \geq i, l \geq j+1 \right\},$$
$$C_{ij} = \left\{ m_{kl} \in M_{n\times(p+1)} : (k-i-1)(s+1)+l-1 \leq j-2, k \geq i+1, l \geq 1 \right\}.$$

In Fig. 1 the entry m_{ij} is the point $*$, the outneighborhood of $*$ is the $\bullet$ points partitioned into the sets A_{ij}, B_{ij} and C_{ij} (the triangles that appear at the top, to the right and to the left respectively), this triangles show how the outneighbors of m_{ij} intersects the outneighbors of 0 and clearly $\mathbf{H}_{p,s}$ is a tournament. Moreover, the matrix entries denoted by **.** are not vertices of $\mathbf{H}_{p,s}$, and the inneighborhood of $*$ consists of the matrix entries $\circ$, since the $\circ$ vertices to the right of A_{ij} have m_{ij} in their C-set, those above C_{ij} have m_{ij} in their B-set, and the others have m_{ij} in their A-set. In the last row, only the first r entries of the matrix are vertices of the tournament $\mathbf{H}_{p,s}$.

Lemma 1. *The tournaments induced by A_{ij}, B_{ij}, C_{ij} are isomorphic to*

$$\mathbf{H}_{1+(i-2)(s+1),s}, \mathbf{H}_{p-(i-1)(s+1)-j,s}, \mathbf{H}_{j-1,s}$$

respectively.

Proof. Let $\Psi_A : A_{ij} \to H_{1+(i-2)(s+1),s}$, $\Psi_B : B_{ij} \to H_{p-(i-1)(s+1)-j,s}$, $\Psi_C : C_{ij} \to H_{j-1,s}$, defined as follows:

$m_{kl} \in A_{ij}$, then $\Psi_A(m_{kl}) = m_{k,l-j+1}$.

$m_{kl} \in B_{ij}$, then $\Psi_B(m_{kl}) = m_{k-i+1,l-j}$

$m_{kl} \in C_{ij}$, then $\Psi_C(m_{kl}) = m_{k-i,l}$.

Clearly Ψ_A, Ψ_B, Ψ_C are all isomorphisms. □

Proposition 2. *The tournament $\mathbf{H}_{p,s}$ is TT_{p+1}-free, moreover, the maximum order of a transitive subtournament in $\mathbf{H}_{p,s}$ is p.*

Proof. The set $\{m_{1j} \in M_{n\times(p+1)} : 1 \le j \le p\}$ is acyclic and has order p, thus the maximum order of a transitive subtournament in $\mathbf{H}_{p,s}$ is at least p.

Remember that $p = (n-1)(s+1) + r$, $n \ge 1$ and $0 < r \le s+1$.

We prove by induction on p, for all $s \ge 0$, $\mathbf{H}_{p,s}$ is TT_{p+1}-free. Fix $s \ge 0$.

If $1 \le p \le s+1$, then $n = 1$ and $p = r$. Therefore the tournament $\mathbf{H}_{p,s}$ is the set $\{m_{1j} \in M_{1\times(p+1)} : 1 \le j \le p\}$ and $\mathbf{H}_{p,s}$ is a transitive tournament of order p.

Now $n \ge 2$ and $p = (n-1)(s+1) + r$, $0 < r \le s+1$.

Let $p = s+2$, then $n = 2$ and $r = 1$. It follows that the tournament $\mathbf{H}_{p,s}$ is isomorphic to the outneighborhood of 0 of the circulant tournament $\vec{C}_{2(p+1)+1}(1, 2, \ldots, p, p+2)$ and this outneighborhood is TT_{p+1}-free [10].

Suppose that all $\mathbf{H}_{k,s}$ are TT_{k+1}-free, for $1 \le k < p$.

Let TT_q be a transitive tournament in $\mathbf{H}_{p,s}$. We prove that $q \le p$.

Let m_{ij} be the source of TT_q. Then $N^+(m_{ij})$ is the disjoint union of transitive subtournaments of A_{ij}, B_{ij}, C_{ij}. Let $T_U = TT_q[U]$, for $U \in \{A_{ij}, B_{ij}, C_{ij}\}$. So

$$V(TT_q) = \{m_{ij}\} \cup V(T_{A_{ij}}) \cup V(T_{B_{ij}}) \cup V(T_{C_{ij}})$$

and since $T_{A_{ij}}, T_{B_{ij}}, T_{C_{ij}}$ are disjoint, then

$$|V(TT_q)| = 1 + |V(T_{A_{ij}})| + |V(T_{B_{ij}})| + |V(T_{C_{ij}})|.$$

By Lemma 1 and by induction hypothesis, it follows that $|V(T_{A_{ij}})| \le 1 + (i-2)(s+1)$, $|V(T_{B_{ij}})| \le p - (i-1)(s+1) - j$, $|V(T_{C_{ij}})| \le j-1$.

Let $i = 1$, then $A_{1,j} = \emptyset$, and

$$|V(TT_q)| \le 1 + 0 + (p-j) + (j-1) = p.$$

Let $i \ge 2$, then

$$\begin{aligned} |V(TT_q)| &\le 1 + (1 + (i-2)(s+1)) + (p - (i-1)(s+1) - j) + (j-1) \\ &= p - s. \end{aligned}$$

Then the maximum order of a transitive subtournament in $\mathbf{H}_{p,s}$ is p. □

Finally we prove the following

Lemma 3. *The tournament induced by $N^+(i)$ is isomorphic to $\mathbf{H}_{p,s}$.*

Proof. Since $\vec{C}_{n(p+1)+1}(J)$ is vertex transitive, we may assume that $i = 0$. Let T be the tournament induced by $N^+(0)$. Note that $N^+(0) = J$.

Let $v \in V(T)$, and $i, j \in \mathbb{N}$ such that $v = (p+1)(i-1) + j$, with $1 \le i \le n$ and $1 \le j \le p+1$. We define the morphism $\psi : V(T) \to H_{p,s}$ with $\psi(v) = m_{ij}$. Note that given any vertex $v \in V(T)$, we calculate i and j as instructed, so ψ is well defined. Clearly ψ is an isomorphism between the subtournament induced by $N^+(0)$ and the tournament $\mathbf{H}_{p,s}$. Note that in Fig. 1 the inneighborhood of 0 are the points **.** and the outneighborhood are the points $\{*, \bullet, \circ\}$. □

From now on, we denote the tournament $\vec{C}_{n(p+1)+1}(J)$ by $\vec{C}_{n(p+1)+1}(H_{p,s})$.

3. Infinite families of $(n+1)$-dichromatic vertex critical circulant tournaments

In this section, we demonstrate that, for $n \geq 2$, $s \geq 0$, and $p = (n-1)(s+1)+1$, the tournament $\vec{C}_{n(p+1)+1}(H_{p,s})$ is a vertex critical circulant tournament with dichromatic number $n+1$. Then the family $\mathcal{H}_n = \{\vec{C}_{n(p+1)+1}(H_{p,s}), s \in \mathbb{N}, p = (n-1)(s+1)+1\}$ is an infinite family of $(n+1)$-dichromatic vertex critical circulant tournaments. Note that for each $s \geq 0$ there is such a tournament in this family. We use Lemma 3 to conclude the properties of the family $\vec{C}_{n(p+1)+1}(H_{p,s})$.

Note that by symmetry (see Fig. 1), it is easy to see that $|\{i, -i\} \cap J| = 1$ for all $i \in \mathbb{Z}_{n(p+1)+1} \setminus \{0\}$, thus $\vec{C}_{n(p+1)+1}(J)$ is a circulant tournament.

Theorem 4. *The maximum order of a transitive subtournament in $\vec{C}_{n(p+1)+1}(H_{p,s})$ is $p+1$.*

Proof. The proof follows by Proposition 2, and the fact that $\{0, 1, , \ldots, p\}$ induces a transitive subtournament of $\vec{C}_{n(p+1)+1}(H_{p,s})$. $\square$

Theorem 5. *$\vec{C}_{n(p+1)+1}(H_{p,s})$ is an $(n+1)$-dichromatic vertex critical circulant tournament.*

Proof. By Theorem 4, the chromatic classes are of order at most $p+1$, thus

$$\begin{aligned} dc(\vec{C}_{n(p+1)+1}(H_{p,s})) &\geq \left\lceil \frac{n(p+1)+1}{p+1} \right\rceil \\ &= n+1. \end{aligned}$$

We define the coloring $\varphi : V(\vec{C}_{n(p+1)+1}(H_{p,s})) \to I_{n+1}$ as follows:

$$\varphi(v) = \begin{cases} k, & \text{if } v \in \{k(p+1) + \{1, 2, \ldots, p+1\}, k \in \{0, 1, \ldots, n-1\}\} \\ n, & \text{if } v = 0. \end{cases}$$

φ is an $(n+1)$-coloring without monochromatic cycles, thus

$$dc(\vec{C}_{n(p+1)+1}(H_{p,s})) = n+1.$$

Moreover, $\vec{C}_{n(p+1)+1}(H_{p,s})$ is vertex critical since it is vertex transitive and φ has a class with only one element. $\square$

Note that, for $n = 2$, the family $\vec{C}_{n(p+1)+1}(H_{p,s})$ is isomorphic to the family

$$\vec{C}_{2m+1}(1, 2, \ldots, m-1, m+1) \text{ [10]},$$

which is 3-dichromatic.

For $n = 3$ the family $\vec{C}_{n(p+1)+1}(H_{p,s})$ is isomorphic to the family

$$D_m = \vec{C}_{6m+1}(1, 2, \ldots, 2m-1, 2m+1, 2m+2, \ldots, 3m, 4m+1) \text{ [7]}$$

which is 4-dichromatic.

If $s = 0$, then $p = n$ and we have a family, which has exactly one $(n+1)$-dichromatic circulant tournament for each $n \geq 2$.

Corollary 6. *$\vec{C}_{n^2+n+1}(H_{n,0})$ is $(n+1)$-dichromatic vertex critical for each $n \geq 2$.*

Note that for $n = 2, 3, 5$ we have the special tournaments ST_7 [3], ST_{13} [11] and ST_{31} [12].

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