On the microscopic nature of dissipative effects in special relativistic kinetic theory

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Abstract

A microscopic formulation of the definition of both the heat flux and the viscous stress tensor is proposed in the framework of kinetic theory for relativistic gases emphasizing on the physical nature of such fluxes. A Lorentz transformation is introduced as the link between the laboratory and local comoving frames and thus between molecular and chaotic velocities. With such transformation, the dissipative effects can be identified as the averages of the chaotic kinetic energy and the momentum flux out of equilibrium, respectively. Within this framework, a kinetic foundation of the ensuing transport equations for the relativistic gas is achieved. To our knowledge, this result is completely novel.

I. INTRODUCTION

Relativistic kinetic theory is not a new subject, however it now finds itself in a spotlight due to the increasing interest in relativistic thermodynamics triggered by recent heavy ion collisions experiments, electron-positron plasma generation, and the traditional astrophysical applications of relativistic hydrodynamics. The theory has its roots in original works by Jüttner [1] for the equilibrium case while the first kinetic theory treatment was formulated by Israel [2]. In such work, the author finds an expression for the stress energy tensor by solving the Boltzmann equation using a Chapman-Enskog expansion. However, since the systematic (hydrodynamic) and chaotic (or peculiar) components of the total velocity of a given molecule are not explicitly distinguished, the different contributions to this tensor cannot be identified in the same fashion as in the non-relativistic case [3, 4]. Instead, projections in parallel and orthogonal directions with respect to the hydrodynamic velocity of the stress energy tensor are used and the interpretations of the different contributions agree with those that follow from the phenomenological counterpart as developed by Eckart [5]. This procedure is essentially followed by most authors [6, 7].

On the other hand, in non-relativistic kinetic theory, a clear distinction can be made between the effects caused by the "bulk", or mechanical, properties of the fluid and its microscopic ones. This permits the identification of dissipative fluxes, i. e. heat and viscosity effects, as averages of chaotic quantities [4]. In particular, the interpretation of heat flux as the average of the chaotic kinetic energy flux, as defined more than a century ago by R. Clausius [8, 9] and J. C. Maxwell [10], is asserted. This concept is absent in the relativistic case, as was clearly noted in Ref. [11]. In that work, a first proposal of how to introduce the chaotic velocity concept in relativistic kinetic theory was put forward. In this work, we follow the same line of thought and take it a step forward by explicitly introducing Lorentz transformations in the stress-

energy tensor integral in order to separate mechanical and chaotic effects. By doing so, we are able to clearly define the heat flux and the viscous stress tensor as the average of chaotic energy and momentum fluxes in an arbitrary frame, respectively.

To accomplish this task we have divided this work as follows. In Section II, we briefly review the non-relativistic setup for calculating the dissipative fluxes. The relativistic framework is introduced in Section III where Lorentz transformations are used in order to introduce the chaotic velocity and obtain the corresponding expressions for the heat flux and viscous stress tensor. A brief discussion of the results and final remarks are included in Section IV.

II. NON-RELATIVISTIC KINETIC THEORY

Kinetic theory serves as the microscopic foundation of irreversible thermodynamics and is capable of producing both the system of transport equations as well as the constitutive equations needed in order to make it a complete set describing the dynamics of fluids [3, 4]. As usual, the distribution function $f(\vec{r}, \vec{v}, t)$ is such that $f(\vec{r}, \vec{v}, t) \, d\vec{r} d\vec{v}$ is the number of molecules contained in a 6-box in the phase space corresponding to position $\vec{r}$ and molecular velocity $\vec{v}$. The local variables are thus defined as averages weighted by this function. The local particle density, hydrodynamic velocity and energy density are thus defined as

$$n = \int f^{(0)} d^3v \tag{1}$$

$$\vec{u} = \frac{1}{n} \int \vec{v} f^{(0)} d^3v \tag{2}$$

$$e = \frac{1}{n} \int \frac{1}{2} m v^2 f^{(0)} d^3v \tag{3}$$

respectively, where $f^{(0)}$ is the local equilibrium distribution function:

$$f^{(0)}\left(\vec{r}, \vec{v}, t\right) = n \left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{m\left(\vec{v} - \vec{u}\right)^2}{2 k_B T}\right) \tag{4}$$

T being the temperature, m the molecular mass and k_B the Boltzmann constant. The evolution of the distribution function is given by the Boltzmann equation. For a simple (one component), non-degenerate, diluted gas in the absence of external fields the kinetic equation reads

$$\frac{df}{dt} = J\left(ff'\right) \tag{5}$$

where, if g and σ are the relative velocity and cross section for a collision between two particles respectively, the collision term is given by

$$J\left(f,\, f'\right) = \int \int \left\{f'f_1' - f f_1\right\} g\sigma d\Omega dv_1^3 \tag{6}$$

Primes denote quantities after the interaction and Ω is the solid angle. The well known Maxwell-Boltzmann distribution function given in Eq. (4) is precisely the solution of $J\left(f,\, f'\right) = 0$, namely the homogeneous Boltzmann equation. The solution to the inhomogeneous, out of equilibrium, case is in general obtained via the Chapman-Enskog method in which the general solution is written as

$$f = f^{(0)} + f^{(1)} \tag{7}$$

where the second term contains corrections to the equilibrium solution to first order in the gradients of the local variables and gives rise to the dissipative fluxes. This term includes only dissipative effects once the solubility constraints

$$\int f^{(1)} d^3v = \int \vec{v} f^{(1)} d^3v = \int v^2 f^{(1)} d^3v = 0 \tag{8}$$

are introduced such that the local variables are defined through the local equilibrium distribution solely.

In this framework, the transport equations are obtained by multiplying Eq. (5) by a collision invariant and integrating over $\vec{v}$. Such procedure yields the Maxwell-Enskog transport equation

$$\frac{\partial}{\partial t} \int \psi f d^3 v + \nabla \cdot \int \psi \vec{v} f d^3 v = 0 \tag{9}$$

which accounts for particle, momentum and energy balances for $\psi = 1$, $\vec{v}$, v^2, respectively. Indeed, taking $\psi = 1$ in Eq. (7) yields the continuity equation

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\vec{u}) = 0 \tag{10}$$

For $\psi = \vec{v}$ one obtains

$$\frac{\partial (n\vec{u})}{\partial t} + \nabla \cdot \overleftrightarrow{\mathcal{T}} = 0 \tag{11}$$

where we introduced the stress tensor

$$\overleftrightarrow{\mathcal{T}} = \int \vec{v}\vec{v} f d^3 v \tag{12}$$

Finally, the energy balance is obtained for $\psi = v^2$:

$$\frac{\partial ne}{\partial t} + \nabla \cdot \vec{J_e} = 0 \tag{13}$$

where we have defined the total energy flux as $\vec{J_e} = \int v^2 \vec{v} f d^3 v$.

In order to isolate the purely dissipative contributions in $\overleftrightarrow{\mathcal{T}}$ and $\vec{J_e}$, one decomposes the molecular velocity in its two basic components, usually written as

$$\vec{v} = \vec{u} + \vec{k} \tag{14}$$

where $\vec{k}$ is the chaotic or peculiar component. In this case such expression arises in a very natural way by observing the argument of the exponential function in Eq. (4). It is clear that $\int \vec{k} f d^3 v = 0$ in view of Eqs. (4) and (10) and thus

$$e = \frac{1}{2} u^2 + \varepsilon \tag{15}$$

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$$\overleftrightarrow{\mathcal{T}} = n\vec{u}\vec{u} + nk_BT\mathbb{I} + \overleftrightarrow{\tau} \tag{16}$$

$$\vec{J}_e = \frac{1}{2}nu^2\vec{u} + n\vec{u}\varepsilon + n\vec{u} \cdot (nk_BT\mathbb{I} + \overleftrightarrow{\tau}) + \vec{q} \tag{17}$$

where

$$n\varepsilon = \int \frac{k^2}{2}f^{(0)}d^3k = \frac{3}{2}k_BT \tag{18}$$

is the internal energy density per particle and the dissipative fluxes are given by

$$\overleftrightarrow{\tau} = \int \vec{k}\vec{k}f^{(1)}d^3k \tag{19}$$

$$\vec{q} = \int \frac{k^2}{2}\vec{k}f^{(1)}d^3k \tag{20}$$

Introducing these definitions in the transport equations and using the local equilibrium assumption, one obtains the well known set of hydrodynamic equations for the non-relativistic fluid.

In the next section, it will be shown how these ideas can be extrapolated in a very natural way to the relativistic framework. In order to make the transition more clear we want to point out at this stage the key role of the transformation $\vec{v} = \vec{u} + \vec{k}$ in the formalism. Notice that such a transformation can also be expressed in terms of a Galilean matrix in space-time, that is

$$v^\mu = \mathcal{G}^{\mu\nu}k_\nu \tag{21}$$

where the Galilean transformation is given by

$$\mathcal{G}^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & u_x/c \\ 0 & 1 & 0 & u_y/c \\ 0 & 0 & 1 & u_z/c \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{22}$$

and

$$v^\mu = \begin{pmatrix} v_x \\ v_y \\ v_z \\ c \end{pmatrix} \qquad k^\mu = \begin{pmatrix} k_x \\ k_y \\ k_z \\ c \end{pmatrix} \tag{23}$$

Whence, the decomposition $\vec{v} = \vec{u} + \vec{k}$ can be viewed as a change in reference frames where an observer comoving with the volume element of the fluid whose hydrodynamic velocity is $\vec{u}$ will measure a given molecule's velocity as $\vec{k}$ while an observer in the laboratory sees the molecule moving at velocity $\vec{v}$ as given by Eq. (21).

III. RELATIVISTIC KINETIC THEORY

In this section we will address the properties of a dilute, neutral, non-degenerate gas within the realms of special relativity. This system is thus described in a Minkowsky space-time whose metric is given by $ds^2 = dx^2 + dy^2 + dz^2 - cdt^2$. For the molecules in this gas, the molecular four-velocity is given by

$$v^\mu = \gamma_w (\vec{w}, c) \tag{24}$$

where

$$\gamma_w \equiv \gamma(w) = \left(1 - \frac{w^2}{c^2}\right)^{-1/2} \tag{25}$$

and $\vec{w}$ is the velocity. The distribution function has the same interpretation as above, being $f(x^\nu, v^\nu) \, d^3x d^3v$ the occupation number of a phase space cell. The special relativistic Boltzmann equation in the absence of external forces is given by

$$v^\alpha f_{,\alpha} = \dot{f} = J(ff') \tag{26}$$

where the collision term is defined as

$$J(f, f') = \int \int \{f' f_1' - f f_1\} F \sigma d\Omega dv_1^* \tag{27}$$

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Here F is an invariant particle flux [7] which plays the role of the relative velocity, σ is the collision cross section and the invariant differential volume in velocity space is $dv^* = \frac{cd^3v}{v^4}$.

Here, as in our previous works, the proposed solution method for the kinetic equation is the Chapman-Enskog procedure to first order in the gradients [12]. As has been shown elsewhere [13], this solution leads to a constitutive equation for the heat flux in terms of gradients of the state variables. This is consistent with Onsager's regression of fluctuations hypothesis and thus predicts no pathological behaviors in the system of hydrodynamic equations [14]. Additionally, this system of equations to first order in the gradients, as predicted by kinetic theory, has been shown to present no causality issues both in the non-relativistic and relativistic cases [14]. In this case, the local equilibrium function is the Jüttner function

$$f^{(0)} = \frac{n}{4\pi c^3 z \mathcal{K}_2\left(\frac{1}{z}\right)} \exp\left(\frac{\mathcal{U}^\beta v_\beta}{zc^2}\right) \tag{28}$$

where $\mathcal{U}^\beta = \gamma_u\left(\vec{u}, c\right)$ is the hydrodynamic four-velocity, $z = k_B T/mc^2$ is the relativistic parameter and $\mathcal{K}_n$ is the n-th order modified Bessel function of the second kind. As in the non-relativistic case, the transport equations are obtained by multiplying Eq. (26) by collision invariants, in this case $\psi = 1, v^\mu$. Indeed, the corresponding transport equation in the absence of external forces is

$$\left[\int v^\alpha \psi f dv^*\right]_{;\alpha} = 0 \tag{29}$$

which yields the continuity equation for $\psi = 1$, the energy-momentum balance equation for $\psi = mv^\beta$; that is, the momentum balance in the absence of external forces for $\beta = 1, 2, 3$ and the energy balance for $\beta = 4$. Equation (29) can be expressed in a more conventional form as a general conservation law for four-flows by defining the particle and stress-energy fluxes as

$$N^\nu = \int v^\nu f dv^* \tag{30}$$

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$$T^{\mu\nu} = m \int v^\mu v^\nu f dv^* \tag{31}$$

respectively. Thus, the transport equations are given by $N^\nu_{;\nu} = 0$ and $T^{\mu\nu}_{;\nu} = 0$. It is then appealing to write the integrals in Eqs. (30) and (31) in terms of systematic and chaotic quantities in order to separate the different contribution to the fluxes, as done in the previous section (see Eqs. (15) to (20)). To accomplish this, an appropriate transformation law has to be assigned in order to introduce the chaotic velocity. It is important to recall at this point that the hydrodynamic velocity is a local equilibrium quantity and is thus only defined in each differential volume where local equilibrium is assumed. If we fix our attention in a single random molecule, we can consider two reference frames, one in the laboratory (S) and one fixed in the volume where the molecule is contained. This second frame $(\bar{S})$, in which the molecules would be seen static on the average, is moving with a speed $\vec{u}$ as seen by an observer fixed in S. Thus, observers in $\bar{S}$ and S would report that the corresponding velocities are given by

$$\bar{v}^\alpha = \gamma_k \left(\vec{k}, c \right) \tag{32}$$

and

$$v^\beta = \mathcal{L}^\beta_\alpha \bar{v}^\alpha = \mathcal{L}^\beta_\alpha K^\alpha \tag{33}$$

respectively. Here $\mathcal{L}^\beta_\alpha$ is a Lorentz boost with velocity $\vec{u}$ and $K^\alpha = \gamma_k \left(\vec{k}, c \right)$ is the chaotic four-velocity [11]. We wish to remind the reader at this point that the contravariant transformation given in Eq. (33) is equivalent to the relativistic velocity addition law.

With the transformation given in Eq. (33), Eqs. (30) and (31) can be written as

$$N^\mu = \mathcal{L}^\mu_\alpha \int K^\alpha f dK^* \tag{34}$$

$$T^{\mu\nu} = m \mathcal{L}^\mu_\alpha \mathcal{L}^\nu_\beta \int K^\alpha K^\beta f dK^* \tag{35}$$

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where use has been made of the fact that, since dv^* is an invariant quantity, $dv^* = dK^*$. Also the equilibrium distribution function given in Eq. (28) can be written in terms of the chaotic speed by use of the invariant $\gamma_k = \mathcal{U}^\beta v_\beta / c^2$ in a similar fashion as in the the non relativistic case where the argument of the Maxwellian is proportional to k^2. These two properties which allow the calculation of integrals in terms of K are verified in the Appendix. In order to obtain a general expression for $T^{\mu\nu}$, we introduce an irreducible decomposition relative to the hydrodynamic four-velocity direction. That is, in this 3+1 representation a second rank tensor can be expressed as [5]

$$T^{\mu\nu} = \tau \mathcal{U}^\mu \mathcal{U}^\nu + \tau^\mu \mathcal{U}^\nu + \tau^\nu \mathcal{U}^\mu + \tau^{\mu\nu} \tag{36}$$

where $\tau^\mu \mathcal{U}_\mu = 0$ and $\tau^{\mu\nu} \mathcal{U}_\nu = 0$. The scalar, first and second rank tensors introduced can be expressed in terms of $T^{\mu\nu}$ as

$$\tau = T^{\mu\nu} \frac{\mathcal{U}_\mu \mathcal{U}_\nu}{c^4} \tag{37}$$

$$\tau^\mu = -\frac{1}{c^2} h^\mu_\alpha T^{\alpha\beta} \mathcal{U}_\beta \tag{38}$$

$$\tau^{\mu\nu} = h^\mu_\alpha h^\nu_\beta T^{\alpha\beta} \tag{39}$$

respectively. Here $h^{\mu\nu} = g^{\mu\nu} + \mathcal{U}^\mu \mathcal{U}^\nu / c^2$ is the well known projector which satisfies $\mathcal{U}_\mu h^\mu_\nu = 0$. It is important to point out in this stage that in the phenomenological treatment, the quantities above are identified as the internal energy, heat flux and stress tensor without a kinetic theory based justification [5]. These definitions are in turn used in most kinetic treatments [2, 4, 6]. It is precisely the aim of this work to deduce, from purely kinetic grounds, that these quantities are indeed related to internal energy, heat flux and stress *interpreted as averages over chaotic velocities* in a similar fashion as in Eqs. (18)-(20).

The scalar τ can be calculated as

$$\tau = m \frac{\mathcal{U}_\mu \mathcal{U}_\nu}{c^4} \int v^\mu v^\nu f dv^* = m \int \gamma_k^2 f dK^* \tag{40}$$

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which is the internal energy per particle. To see that this is so, consider Eq. (29) with $\psi = mv^4$

$$\frac{\partial}{\partial t}\left(m\int v^4 v^4 f dv^*\right) + \frac{\partial}{\partial x^\ell}\left(m\int v^4 v^\ell f dv^*\right) = 0 \tag{41}$$

where here, as in the rest of this work, latin indices run from 1 to 3 only. It is clear from Eq. (41) that the integral in the first term is indeed the total energy while the second integral is the energy flux. Thus, the equivalent to the total energy moment calculated in a rest frame yields the internal energy only, that is

$$n\varepsilon = mc^2 \int \gamma_k^2 f dK^* \tag{42}$$

and thus,

$$\tau = \frac{n\varepsilon}{c^2} = nm\left(3z + \frac{\mathcal{K}_3\left(\frac{1}{z}\right)}{\mathcal{K}_2\left(\frac{1}{z}\right)}\right) \tag{43}$$

For the vector quantity τ^μ we have

$$\tau^\mu = -\frac{1}{c^2}h_\alpha^\mu \mathcal{U}_\beta \int v^\alpha v^\beta f dv^* \tag{44}$$

which, using again the fact that $\mathcal{U}_\beta v^\beta = -c^2\gamma_k$ can be expressed as an integral over the chaotic velocities as follows

$$\tau^\mu = h_\alpha^\mu \mathcal{L}_\beta^\alpha \int \gamma_k K^\beta f dK^* \tag{45}$$

It can be shown (see the Appendix) that the contraction of the projector with the Lorentz transformation yields a tensor $\mathcal{R}_\nu^\mu = h_\alpha^\mu \mathcal{L}_\nu^\alpha$ given by

$$\mathcal{R}_4^\mu = 0 \tag{46}$$

$$\mathcal{R}_a^\mu = \mathcal{L}_a^\mu \quad \text{for } a = 1, 2, 3 \tag{47}$$

and thus

$$\tau^\mu = \mathcal{R}_\beta^\mu \int \gamma_k K^\beta f dK^* \tag{48}$$

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We now introduce the Chapman-Enskog expansion

$$\tau^\mu = \mathcal{R}^\mu_\beta \int \gamma_k K^\beta f^{(0)} dK^* + \mathcal{R}^\mu_\beta \int \gamma_k K^\beta f^{(1)} dK^* \tag{49}$$

and notice that the first terms vanishes since, for $\beta = 1, 2, 3$ the integral $\mathcal{R}^\mu_\beta \int \gamma_k K^\beta f^{(0)} dK^*$ is odd in k and the $\beta = 4$ term in the sum is zero because $\mathcal{R}^\mu_4 = 0$ for any μ. Thus, only the integral with $f^{(1)}$ survives and we can write

$$\tau^\mu = \mathcal{R}^\mu_\beta \int \gamma_k K^\beta f^{(1)} dK^* \tag{50}$$

In order to re-introduce the Lorentz transformation, we notice that

$$\int \gamma_k K^4 f^{(1)} dK^* = 0 \tag{51}$$

since the internal energy, as all state variables, is obtained only through the equilibrium solution. That is, the subsiadiary condition, which the solution $f^{(1)}$ will be enforced to satisfy, requires

$$\int \gamma_k^2 f^{(i)} dK^* = 0 \qquad \text{for } i \neq 0 \tag{52}$$

Using Eq. (51), one can write Eq. (50) back in terms of $\mathcal{L}^\mu_\beta$ which yields

$$\tau^\mu = \mathcal{L}^\mu_\beta \int \gamma_k K^\beta f^{(1)} dK^* \tag{53}$$

By inspection of Eq. (41) one concludes that the integral $\int \gamma_k K^b f^{(1)} dK^*$ is the heat flux in a rest frame where $v^\alpha = K^\alpha$, and thus

$$q^\beta_{[0]} = c^2 \int \gamma_k K^\beta f^{(1)} dK^* \tag{54}$$

This expression is analogous to the one found in the non-relativistic case and full of physical content. The heat flux is physically the average flux of the chaotic energy, and Eq. (54) is completely consistent which this idea. Now, in an arbitrary frame

$$\tau^\mu = \frac{1}{c^2} \mathcal{L}^\mu_\nu q^\nu_{[0]} \tag{55}$$

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which is, to the authors' knowledge, the first time that the heat flux is obtained only from a kinetic theory standpoint as the average of the peculiar kinetic energy flux of the molecules.

For the second rank tensor in the stress-energy tensor decomposition, we calculate from Eq. (39)

$$\tau^{\mu\nu} = m h^\mu_\alpha h^\nu_\beta \mathcal{L}^\alpha_\eta \mathcal{L}^\beta_\delta \int K^\eta K^\delta f \, dK^* \tag{56}$$

or

$$\tau^{\mu\nu} = m \mathcal{R}^\mu_\eta \mathcal{R}^\nu_\delta \int K^\eta K^\delta \left(f^{(0)} + f^{(1)} \right) dK^* \tag{57}$$

For the local-equilibrium term we have

$$m \mathcal{R}^\mu_\eta \mathcal{R}^\nu_\delta \int K^\eta K^\delta f^{(0)} dK^* = m \mathcal{R}^\mu_a \mathcal{R}^\nu_b \int K^a K^b f^{(0)} dK^* \tag{58}$$

Since $f^{(0)}$ is even in k, only the $a = b$ terms survive and thus

$$m \mathcal{R}^\mu_\eta \mathcal{R}^\nu_\delta \int K^\eta K^\delta f^{(0)} dK^* = p h^{\mu\nu} \tag{59}$$

where we have introduced the well known result for the hydrostatic pressure

$$p = m \int \left(K^1 \right)^2 f^{(0)} dK^* = m \int \left(K^2 \right)^2 f^{(0)} dK^* = m \int \left(K^3 \right)^2 f^{(0)} dK^* \tag{60}$$

and

$$p = n k_B T \tag{61}$$

together with the identity

$$\mathcal{R}^\mu_a \mathcal{R}^\nu_b \delta^{ab} = \mathcal{L}^\mu_a \mathcal{L}^\nu_b \delta^{ab} = h^{\mu\nu} \tag{62}$$

Equation (59) was obtained in a similar fashion (using Lorentz transformations) by Weinberg [15], nevertheless he did not address the dissipative case following a kinetic theory approach.

For the dissipative term in Eq. (57), which we write as $\Pi^{\mu\nu}$, we have

$$\Pi^{\mu\nu} = m\mathcal{R}_\eta^\mu \mathcal{R}_\delta^\nu \int K^\eta K^\delta f^{(1)} dK^* = m\mathcal{L}_a^\mu \mathcal{L}_b^\nu \int K^a K^b f^{(1)} dK^* \qquad (63)$$

If $\Pi_{[0]}^{\alpha\beta}$ is the Navier-Newton tensor calculated in a frame where the fluid is at rest

$$\Pi_{[0]}^{\mu\nu} = mh_\alpha^\mu h_\beta^\nu \int K^\alpha K^\beta f dK^* = m\delta_a^\mu \delta_b^\nu \int K^a K^b f dK^*$$

since in such frame $h_4^\mu = 0$ and $h_a^\mu = \delta_a^\mu$. Thus, the second rank tensor introduced in the stress-energy tensor is

$$\tau^{\mu\nu} = ph^{\mu\nu} + \Pi^{\mu\nu}$$

where

$$\Pi^{\mu\nu} = \mathcal{L}_\alpha^\mu \mathcal{L}_\beta^\nu \Pi_{[0]}^{\alpha\beta} \qquad (64)$$

is the Navier-Newton tensor in an arbitrary frame.

IV. SUMMARY AND FINAL REMARKS

In the previous section, the different contributions to the stress-energy tensor for a single component, dilute gas in the framework of special relativity have been calculated by separating hydrodynamic and chaotic contributions to the molecular velocities. This has been accomplished by introducing Lorentz transformations to relate the velocity of a molecule as measured by an arbitrary observer with the one measured within a differential volume moving at the corresponding hydrodynamic velocity, an idea introduced by two of us in Ref. [11], combined with Eckart's decomposition [5].

The main results of this work can be summarized in the fact that all quantities appearing in Eq. (36) have been obtained strictly from kinetic theory *using the concept of chaotic velocity.* The first two terms are the equilibrium parts of the stress-energy tensor and are well known. The main accomplishment of the calculation here

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shown are the dissipative terms which appear here in a natural way as averages over kinetic energy and momentum fluxes once the transformation between molecular and peculiar velocities is introduced. Also it has been shown that the heat flux transforms as a first rank tensor.

A kinetic derivation of the stress-energy tensor for a dissipative fluid from first principles in kinetic theory has been lacking for some time and thus hindering a clear derivation of the relativistic Navier-Stokes equations. Equations (34) and (35) satisfy both needs and, in turn, pose a new question. Since both heat and momentum fluxes in Eq. (36) are given by Eqs. (55) and (64) respectively, the hydrodynamic velocity factors introduced by the Lorentz transformations will induce new non-linearities in the system of hydrodynamic equations. This could yield new relativistic effects for the relativistic gas which may be measurable. This question and will be addressed in the future.

Appendix

In this appendix the relations

$$dv^* = dK^* \tag{65}$$

and

$$\mathcal{U}^\nu v_\nu = \gamma_k \tag{66}$$

are shown to hold where $\mathcal{U}^\mu$, v^μ and K^μ are the hydrodynamic, molecular and chaotic four-velocities respectively. Also, we verify that the tensor quantity $\mathcal{R}^\mu_\nu$ is indeed given by Eqs. (46) and (47).

To verify Eqs. (65) and (66), we consider two reference frames S and $\bar{S}$ with a relative speed $\vec{u}$ with respect to each other. That is, S may be considered the laboratory frame while $\bar{S}$ is a frame fixed to a volume element in the fluid. For the

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sake of simplicity, we take the x direction parallel to $\vec{u}$. In this situation, we have three four-vectors related to a given molecule in such fluid element

$$K^{\nu} = \gamma_k \left(\vec{k}, c \right) \qquad \text{velocity of the molecule as measured by an observer in } \bar{S}$$

$$v^{\nu} = \gamma_v \left(\vec{w}, c \right) \qquad \text{velocity of the molecule as measured by an observer in } S$$

$$\mathcal{U}^{\nu} = \gamma_u \left(u, 0, 0, c \right) \qquad \text{relative velocity between } S \text{ and } \bar{S}$$

The relationship between tensors in both references frames given by the Lorentz transformation

$$\mathcal{L}^{\mu}_{\nu} = \begin{bmatrix} \gamma_u & 0 & 0 & \frac{u}{c}\gamma_u \\ 0 & 1 & 0 & 0 \\ \frac{u}{c}\gamma_u & 0 & 1 & \gamma_u \end{bmatrix} \tag{67}$$

then

$$A^{\mu} = \mathcal{L}^{\mu}_{\nu} \bar{A}^{\nu} \tag{68}$$

Since the molecule's velocity, as measured in $\bar{S}$, is $\bar{v}^{\nu} = K^{\nu}$ we have

$$v^{\mu} = \mathcal{L}^{\mu}_{\nu} K^{\nu} = \gamma_k \left(\gamma_u \left(u + k_1 \right), k_2, k_3, \gamma_u c \left(1 + \frac{u k_1}{c^2} \right) \right) \tag{69}$$

In order to show the invariance of the volume element $dv^* = c d^3 v / v^4$ we start from

$$d^3 v = J d^3 K \tag{70}$$

where the Jacobian is given by

$$J = det \left[\frac{\partial v^a}{\partial K^b} \right] \tag{71}$$

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and is calculated as follows

$$\frac{\partial v^a}{\partial K^b} = \frac{\partial}{\partial K^b}\left[\mathcal{L}_\nu^a K^\nu\right] = \mathcal{L}_\nu^a \frac{\partial}{\partial K^b}\left[K^\nu\right] = \begin{cases} \gamma_u\left(\delta_b^1 - \frac{u}{c}\frac{K_1}{K_4}\right) & a = 1 \\ \\ \delta_b^a & a \neq 1 \end{cases}$$

where use has been made of the fact that, since K^μ is a four-velocity, $K^\mu K_\mu = -c^2$ and thus

$$0 = K_\mu \frac{\partial K^\mu}{\partial K^b} = K_4 \frac{\partial K^4}{\partial K^b} + K_a \frac{\partial K^a}{\partial K^b} = K_4 \frac{\partial K^4}{\partial K^b} + K_b \tag{72}$$

Then, the Jacobian is

$$J = \gamma_u \left(1 - \frac{u}{c}\frac{K_1}{K_4}\right) = \frac{1}{K_4}\gamma_u\left(K_4 - \frac{u}{c}K_1\right) = \frac{1}{K^4}\gamma_u\gamma_k\left(c + \frac{u}{c}k_1\right) = v^4 \tag{73}$$

and thus

$$\frac{d^3v}{v^4} = \frac{d^3K}{K^4} \tag{74}$$

Regarding the scalar product $\mathcal{U}^\nu v_\nu$ we have

$$\mathcal{U}_\nu v^\nu = \mathcal{U}_\nu \mathcal{L}_\mu^\nu K^\mu \tag{75}$$

which can be readily calculated using that

$$\mathcal{U}_\nu = \gamma_u\left(u, 0, 0, -c\right) \tag{76}$$

and the transformation given in Eq. (67) as follows

$$\mathcal{U}_\nu v^\nu = \gamma_u u \mathcal{L}_\mu^1 K^\mu - c\gamma_u \mathcal{L}_\mu^4 K^\mu = \gamma_u^2 \left[uK^1 + \frac{u^2}{c}K^4 - uK^1 - cK^4\right] = \gamma_u^2 \left[\frac{u^2}{c^2} - 1\right] cK^4 \tag{77}$$

and thus

$$\mathcal{U}_\nu v^\nu = -cK^4 \tag{78}$$

The results in Eqs. (74) and (78) allow the calculation of moments of the distribution function in terms of the chaotic velocity in as similar way as in the non-relativistic case:

$$\int \exp\left(\frac{\mathcal{U}^\beta v_\beta}{zc^2}\right)\mathcal{J}dv^* = \int \exp\left(-\frac{\gamma_k}{c}\right)\mathcal{J}dK^* \tag{79}$$

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where $\mathcal{J}$ is an arbitrary tensor.

Now we turn to the proof of Eqs. (46) and (47). Firstly, since $\mathcal{L}_4^\alpha = \frac{\mathcal{U}^\alpha}{c}$,

$$\mathcal{R}_4^\mu = h_\alpha^\mu \mathcal{L}_4^\alpha = h_\alpha^\mu \frac{\mathcal{U}^\alpha}{c} = 0 \tag{80}$$

To obtain Eq. (47), we separate two cases. For $\mu = 4$, since $h_4^4 = 1 - \gamma^2$ and $h_b^4 = \gamma \frac{\mathcal{U}_b}{c}$

$$\mathcal{R}_a^4 = h_\alpha^4 \mathcal{L}_a^\alpha = \left(1 - \gamma^2\right) \frac{\mathcal{U}_a}{c} + \gamma \frac{\mathcal{U}_b}{c} \mathcal{L}_a^b \tag{81}$$

For the second term we use that

$$\frac{\mathcal{U}_b}{c} \mathcal{L}_a^b = \frac{\mathcal{U}_b}{c} \left(\delta_a^b + \frac{\mathcal{U}_a \mathcal{U}^b}{c^2 (\gamma + 1)}\right) = \gamma \frac{\mathcal{U}_a}{c} \tag{82}$$

and thus

$$\mathcal{R}_a^4 = \frac{\mathcal{U}_a}{c} = \mathcal{L}_a^4 \tag{83}$$

Finally, for $\mu = \ell = 1, 2, 3$

$$\mathcal{R}_a^\ell = h_\alpha^\ell \mathcal{L}_a^\alpha = h_b^\ell \mathcal{L}_a^b + h_4^\ell \mathcal{L}_a^4 \tag{84}$$

or, using that $h_4^\ell = -\gamma \frac{\mathcal{U}^\ell}{c}$

$$\mathcal{R}_a^\ell = \left(\delta_b^\ell + \frac{\mathcal{U}_b \mathcal{U}^\ell}{c^2}\right) \mathcal{L}_a^b - \gamma \frac{\mathcal{U}^\ell \mathcal{U}_a}{c^2} \tag{85}$$

Now, by introducing Eq. (82) in Eq. (85) one obtains

$$\mathcal{R}_a^\ell = \mathcal{L}_a^\ell \tag{86}$$

This completes the proof.

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