# Mixed Cages: monotony, connectivity and upper bounds * 

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#### Abstract

A $[z, r ; g]$-mixed cage is a mixed graph $z$-regular by arcs, $r$-regular by edges, with girth $g$ and minimum order. Let $n[z, r ; g]$ denote the order of a $[z, r ; g]$-mixed cage.

In this paper we prove that $n[z, r ; g]$ is a monotonicity function, with respect of $g$, for $z \in\{1,2\}$, and we use it to prove that the underlying graph of a $[z, r ; g]$-mixed cage is 2 -connected, for $z \in\{1,2\}$. We also prove that $[z, r ; g]$-mixed cages are strong connected. We present bounds of $n[z, r ; g]$ and constructions of $[z, r ; 5]$-mixed graphs and show a $[10,3 ; 5]$-mixed cage of order 50 .

Keywords: Mixed cages, monotonicity, connectivity, projective planes, cages and directed cages.


## 1 Introduction

In this paper we consider graphs which are finite and mixed, that is, they may contain (directed) arcs as well as (undirected) edges. We don't allow multiple edges and arcs.

The mixed regular graphs were introduced in [5]. A mixed regular graph is a simple and finite graph $G$, such that for every $v \in V(G), v$ is the head of $z$ arcs, the tail of $z$ arcs and is incident with $r$ edges. The directed degree of a vertex $v$ is equal to $z$, while the undirected degree is equal to $r$. We set $d=z+r$ to be the degree of $v$. We will consider walks of the form $\left(v_{0}, \ldots, v_{n}\right)$, where eihter $v_{i} v_{i+1}$ is an edge of $G$ or $\left(v_{i}, v_{i+1}\right)$ is an arc of $G$, for $i \in\{0, \ldots, n-1\}$. In other words, the walks could contain edges and arcs, provided that all the arcs are traversed in the same direction. The girth of $G$ is the length of the shortest cycle of $G$, we denote the length of a cycle $C$ as $\ell(C)$. The distance between two vertices $u$ and $v$ denoted by $d(u, v)$ is defined as the shortest length of all $u v$-paths. If $G$ has girth equal to $g$, then $G$ is a $[z, r ; g]$-mixed graph of directed degree $z$, undirected degree $r$ and girth $g$. A $[z, r ; g]$-mixed cage is a $[z, r ; g]$-mixed graph of minimum order. Through this paper we use $n[z, k ; g]$ to denote the order of a $[z, r ; g]$-mixed graph.

The Cage Problem is to find the smallest number, $n(k, g)$, of vertices for a $k$-regular graph of girth $g$. It has been widely studied since cages were introduced by Tutte [14 in 1947 and after Erdös and Sachs [10] proved, in 1963, their existence. A complete survey about this topic and its relevance can be found in [11. Moreover, there exists a lot of results, related with this problem that studied structural properties of cages as monotonicity and connectivity. For instance, Fu, Huang and Rodger [12 proved that if $k \geq 2$ and $3 \leq g_{1}<g_{2}$, then $n\left(k, g_{1}\right)<n\left(k, g_{2}\right)$. Concerning the connectivity, it is known that if $G$ is a $(k, g)$-cage, then $\lambda(G)=k$ [13, 15], and for every odd girth $g \geq 7, \kappa(G) \geq\lfloor k / 2\rfloor+1$ [6]. For even girth $g \geq 6,(k, g)$-cages with $k \geq 3$ are $(t+1)$-connected, $t$ being the largest integer such that $t^{2}+2 t^{2} \leq k[7$.

In this paper we are interested in the Mixed Cage Problem, that is, find constructions of $[z, r ; g]$-mixed regular graphs, with specified degrees $z, r$, girth $g$ and minimum order. The work concerning constructions

[^0]of $[z, r ; 5]$-mixed cages starts in [5] and continue in [2] where the authors give constructions of small $[z, r ; 5]$ mixed cages with similar techniques than used in [1] to construct small regular graphs of girth five.

The paper is organized as follows: in Section 2 we study structural properties of mixed cages. We prove that $n[z, r ; g]$ is a monotonicity function with respect of $g$, for $z \in\{1,2\}$. As a consequence of this result we show that the underlying graph of a $[z, r ; g]$-mixed cages is 2 -connected, for $z \in\{1,2\}$. We also show that every $[z, r ; g]$-mixed cage is strong connected.

In Section 3, we present a lower bound for $n[z, r ; g]$, specifically we show that if $g \geq 5$, then $n[z, r ; g] \geq$ $n_{0}(r, g)+2 z$, where $n_{0}(r, g)$ is the More bound. We give two different constructions that provide us new upper bounds for $n[z, k ; g]$. In the first construction we use the incident finite graph of a partial plane defined over any finite field generated by a prime. This construction was also used in previous papers, for example, to construct regular graphs of girth 5 [3] and to construct Mixed Moore graphs of diameter 2 [4]. With this construction we state that $n[z, r ; 5] \leq 10 z r$. In the second construction, we establish that $n[z, r ; g] \leq g n_{0}(r, g)$. In particular, we construct a $(10,3 ; 5)$-mixed graph of order 50 , that results on a ( 10,$3 ; 5$ )-mixed cage.

## 2 Monotonicity and connectivity

This section is divided into two parts. In the first one we focus on the study of the monotonicity. In the second part we study the connectivity of mixed cages.

### 2.1 Monotonicity

Let $G$ be a mixed graph. Let $E^{*}(G)=A(G) \cup E(G)$ and, if $v$ is a vertex of $G$, let $N^{*}(v)=N(v) \cup N^{+}(v) \cup$ $N^{-}(v)$

Lemma 1. Let $z \in\{1,2\}$. Every $[z, r ; g]$-mixed cage contains a cycle of length $g$ with either two consecutive arcs or two consecutive edges.

Proof. Let $G$ be a $[z, r ; g]$-mixed cage with $z \in\{1,2\}$. Suppose that every cycle of $G$ of length $g$ is alternating by arcs and edges. Let $C$ be a cycle of length $g$. Let $\overrightarrow{x u} \in A(C)$ and $u v \in E(C)$. Since $C$ is an induced cycle, there exists an arc $\overrightarrow{u y}$ with $y \in V(G) \backslash V(C)$. Let $N(y)=\left\{v_{1}, \ldots, v_{r}\right\}$ and $u^{\prime} \in N^{+}(y)$. If $w \in N(y) \cup N^{+}(y)$, then $w \notin V(C)$. Otherwise, $G$ would contains a cycle of length $g$ with either two consecutive arcs or with two consecutive edges, a contradiction.

We divide the proof into two cases. In each case we construct a $[z, r ; g]$-mixed graph with less vertices than $G$, giving a contradiction.

Case 1) Suppose $z=1$. If $r$ is even, let $E^{\prime}=\left\{v_{2 i-1} v_{2 i}: 1 \leq i \leq r / 2\right\} \cup\left\{\overrightarrow{u u^{\prime}}\right\}$. We define $G^{\prime}$ as $G^{\prime}=G-y+E^{\prime}$ (see Figure 1). Observe that $G^{\prime}$ is a mixed graph 1-regular in arcs and $r$-regular in edges. Moreover, the cycle $C$ is totally contained in $G^{\prime}$. Hence, $G^{\prime}$ is a $\left[1, r ; g^{\prime}\right]$-mixed graph with $g^{\prime}=g\left(G^{\prime}\right) \leq g$. Let $C^{\prime}$ be a cycle of $G^{\prime}$ such that $\ell\left(C^{\prime}\right)=g^{\prime}$.

If $\left|E^{*}\left(C^{\prime}\right) \cap E^{\prime}\right|=0$, then $C^{\prime}$ is totally contained in $G$. Thus, $g \leq \ell(C)=g^{\prime}$, a contradiction. Suppose that $\left|E^{*}\left(C^{\prime}\right) \cap E^{\prime}\right|=1$. Let $\left\{\alpha_{i} \alpha_{j}\right\}=E^{*}\left(C^{\prime}\right) \cap E^{\prime}$. Observe that $C^{\prime}-\alpha_{i} \alpha_{j}$ is totally contained in $G$. Hence, either $\ell\left(C^{\prime}\right) \geq d_{G-y}\left(\alpha_{i}, \alpha_{j}\right)+1 \geq g$ or $\ell\left(C^{\prime}\right) \geq d_{G-y}\left(\alpha_{j}, \alpha_{i}\right)+1 \geq g$, giving a contradiction. Continue assuming that $\left|E\left(C^{\prime}\right) \cap E^{\prime}\right| \geq 2$. Since $E^{\prime}$ is an independent set of arcs and edges, there exist $e_{1}, e_{2} \in E^{*}(C) \cap E^{\prime}$ such that $C^{\prime}$ contains an $\alpha_{i} \alpha_{j}$-path totally contained in $G-y$. Therefore, $\ell\left(C^{\prime}\right) \geq d_{G-y}\left(\alpha_{i}, \alpha_{j}\right)+2>g$, a contradiction.

If $r$ is odd, let $w=v_{r}, N(w)=\left\{v_{1}^{\prime}, \ldots, v_{r}^{\prime}\right\}$, where $v_{r}^{\prime}=y, N^{-}(w)=\left\{x^{\prime}\right\}$ and $N^{+}(w)=\left\{y^{\prime}\right\}$. Let $E^{\prime}=\left\{v_{2 i-1} v_{2 i}, v_{2 i-1}^{\prime} v_{2 i}^{\prime}: 1 \leq i \leq(r-1) / 2\right\} \cup\left\{\overrightarrow{u u^{\prime}}, \overrightarrow{x^{\prime} y^{\prime}}\right\}$. Define the mixed graph $G^{\prime}$ as $G^{\prime}=G-\{y, w\}+E^{\prime}$. By a similar analysis to the previous case, we conclude that $g(G)=g$. Therefore, $G^{\prime}$ is a $[1, r ; g]$-mixed graph with two vertices less than $G$, yielding a contradiction.

Case 2) Suppose $z=2$. Let $N^{-}(y)=\{u, s\}$ and $N^{+}(y)=\left\{u^{\prime}, s^{\prime}\right\}$. If $r$ is even, let $E^{\prime}=\left\{v_{1} v_{2}, v_{3} v_{4}, \ldots\right.$, $\left.v_{r-1} v_{r}\right\}$. Since $d^{-}\left(u^{\prime}\right)=d^{-}\left(s^{\prime}\right)=2$, it follows that $\left|N^{-}\left(u^{\prime}\right) \cap\{u, s\}\right| \leq 1$ and $\left|N^{-}\left(s^{\prime}\right) \cap\{u, s\}\right| \leq 1$. Next,


Figure 1: Construction of $G^{\prime}$ from a $[1, r ; g]$-mixed cage with $r$ even.
we define a set $A^{\prime}$ depending on the the sets $N^{-}\left(u^{\prime}\right)$ and $N^{-}\left(s^{\prime}\right)$. If $u \in N^{-}\left(u^{\prime}\right)$ or $s \in N^{-}\left(s^{\prime}\right)$, then $A^{\prime}=\left\{\overrightarrow{u s^{\prime}}, \overrightarrow{s u^{\prime}}\right\}$. In other case, $A^{\prime}=\left\{\overrightarrow{u u^{\prime}}, \overrightarrow{s s^{\prime}}\right\}$. Define $G^{\prime}$ as $G^{\prime}=G-y+E^{\prime}+A^{\prime}$ (see Figure 2). By a similar analysis to that of Case 1), $G^{\prime}$ is a $[2, r ; g]$-mixed graph with less order than $G$, a contradiction.


Figure 2: Construction of $G^{\prime}$ from a $[2, r ; g]$-mixed cage with $r$ even and $\overrightarrow{s u^{\prime}} \in A(G)$.
If $r$ is odd, let $w=v_{r}, N(w)=\left\{v_{1}^{\prime}, \ldots, v_{r}^{\prime}\right\}$, where $v_{r}^{\prime}=y, N^{-}(w)=\left\{x^{\prime}, x^{\prime \prime}\right\}$ and $N^{+}(w)=\left\{y^{\prime}, y^{\prime \prime}\right\}$. Let $E^{\prime}=\left\{v_{2 i-1} v_{2 i}, v_{2 i-1}^{\prime} v_{2 i}^{\prime}: 1 \leq i \leq(r-1) / 2\right\}$. Since $d^{-}\left(u^{\prime}\right)=d^{-}\left(s^{\prime}\right)=2$, it follows that $\left|N^{-}\left(u^{\prime}\right) \cap\{u, s\}\right| \leq 1$ and $\left|N^{-}\left(s^{\prime}\right) \cap\{u, s\}\right| \leq 1$. Define a set $A_{y}$ depending on the sets $N^{-}\left(u^{\prime}\right)$ and $N^{-}\left(s^{\prime}\right)$. If $u \in N^{-}\left(u^{\prime}\right)$ or $s \in N^{-}\left(s^{\prime}\right)$, then $A_{y}=\left\{\overrightarrow{s u^{\prime}}, \overrightarrow{u s^{\prime}}\right\}$. In other case, $A_{y}=\left\{\overrightarrow{u u^{\prime}}, \overrightarrow{s s^{\prime}}\right\}$. Analogously, we define $A_{w}$ depending on the sets $N^{-}\left(y^{\prime}\right)$ and $N^{-}\left(y^{\prime \prime}\right)$. If $x^{\prime} \in N^{-}\left(y^{\prime}\right)$ or $x^{\prime \prime} \in N^{-}\left(y^{\prime \prime}\right)$, then $A_{w}=\left\{\overrightarrow{x^{\prime} y^{\prime \prime}}, \overrightarrow{x^{\prime \prime} y^{\prime}}\right\}$. In other case, $A_{w}=\left\{\overrightarrow{x^{\prime} y^{\prime}}, \overrightarrow{x^{\prime \prime} y^{\prime \prime}}\right\}$.

Define $G^{\prime}$ as $G^{\prime}=G-\{y, w\}+E^{\prime}+A_{y}+A_{w}$. Again, by a similar analysis to that of Case 1), a contradiction is obtained.

Therefore, every $[z, r ; g]$-mixed cage have at less one cycle of length $g$ with two consecutive arcs or two consecutive edges.

In [5], Araujo-Pardo, Hernández-Cruz and Montellano-Ballesteros, calculated a general lower bound for a $[z, r ; g]$-mixed cage in the following theorem:
Theorem 2. If $n[1, r ; g]$ is the order of $a[1, r ; g]$-mixed cage, then

$$
n[1, r ; g] \geq n_{0}[1, r ; g]= \begin{cases}2\left(1+\sum_{i=1}^{(g-3) / 2} n_{0}(r, 2 i+1)\right)+n_{0}(r, g) & \text { if } g \text { is odd } \\ 2\left(1+\sum_{i=1}^{(g-2) / 2} n_{0}(r, 2 i+1)\right) & \text { if } g \text { is even }\end{cases}
$$

Now, we can prove the main theorem of this section.
Theorem 3. Let $z \in\{1,2\}, r \geq 1$ and $3 \leq g_{1}<g_{2}$ be integers, then

$$
n\left[z, r ; g_{1}\right]<n\left[z, r ; g_{2}\right]
$$

Proof. It suffices to show that if $z \in\{1,2\}, r \geq 1$ and $g \geq 3$, then $n[z, r ; g]<n[z, r ; g+1]$. Let $G$ be a $[z, r ; g+1]$-mixed cage. Let $C$ be a cycle of $G$ such that $\ell(C)=g+1$. By Lemma 1, $C$ contains two consecutive arcs or two consecutive edges. Let $u \in V(C)$. Suppose that $N(u)=\left\{v_{1}, \ldots, v_{r}\right\}, x_{1} \in N^{-}(u)$ and $y_{1} \in N^{+}(u)$.

We divide the proof in cases depending on the value of $z$ and the parity of $r$. However, the general reasoning for all cases is the same: from the graph $G$, by deleting a set of vertices and adding a set of arcs and a set of edges, a $\left[z, r ; g^{\prime}\right]$-mixed graph $G^{\prime}$ with girth $g^{\prime}<g+1$ and $\left|V\left(G^{\prime}\right)\right|<n[z, r ; g]$ is constructed.

Case 1) Suppose $z=1$. If $g=3$, by Theorem $2, n[1, r ; 3]=2+n_{0}(r, 3)=r+3, n[1, r ; 4] \geq 2\left(1+n_{0}(r, 3)\right)=$ $2 r+4$, and the result follows. Continue assuming $g \geq 4$.

Case 1.1) Suppose that $r$ is even. If $C$ has two consecutive edges $v_{1} u$ and $u v_{2}$ (see Figure 3), let $E^{\prime}=\left\{v_{2 i-1} v_{2 i}: 1 \leq i \leq r / 2\right\} \cup\left\{\overrightarrow{x_{1} y_{1}}\right\}$.


Figure 3: Operation in a $[1, r ; g+1]$-mixed cage with $r$ even, in a cycle with two edges consecutive.
Let $G^{\prime}=G-u+E^{\prime}$. Observe that $g\left(G^{\prime}\right) \leq g$, since $G^{\prime}$ contains the cycle $C-u+v_{1} v_{2}$. We claim that $g\left(G^{\prime}\right)=g$. Let $C^{\prime}$ be a cycle of $G^{\prime}$ such that $\ell\left(C^{\prime}\right)=g\left(G^{\prime}\right) \leq g$. If $E\left(C^{\prime}\right) \cap E^{\prime}=\emptyset$, then $E^{*}\left(C^{\prime}\right) \subseteq E(G)$, implying that $\ell\left(C^{\prime}\right) \geq g+1$, a contradiction. Hence, $E^{*}\left(C^{\prime}\right) \cap E^{\prime} \neq \emptyset$. If $\left|E^{*}\left(C^{\prime}\right) \cap E^{\prime}\right|=1$, then $E^{*}\left(C^{\prime}\right) \cap E^{\prime}=\left\{\alpha_{i} \alpha_{j}\right\}$, which implies that $C^{\prime}-\alpha_{i} \alpha_{j}$ is an $\alpha_{i} \alpha_{j}$-path or an $\alpha_{j} \alpha_{i}$-path totally contained in $G$ of length at least $g-1$. Therefore, $g \leq d_{G-u}\left(\alpha_{i}, \alpha_{j}\right)+1 \leq \ell\left(C^{\prime}\right)=g\left(G^{\prime}\right) \leq g$. Suppose that $\left|E^{*}\left(C^{\prime}\right) \cap E^{\prime}\right| \geq 2$. Since $E^{\prime}$ is an independent set of edges and arcs, there exist $e_{1}, e_{2} \in E^{*}\left(C^{\prime}\right) \cap E^{\prime}$ such that there is an $\alpha_{i} \alpha_{j}$-path in $C^{\prime}$ totally contained in $G-u$, where $\alpha_{i}$ is a vertex of $e_{1}$ and $\alpha_{j}$ is a vertex of $e_{2}$. Since $\alpha_{i}, \alpha_{j} \in N^{*}(u)$, the length of every $\alpha_{i} \alpha_{j}$-path in $G-u$ is at least $g-1$. Hence, $g<d_{G-u}\left(\alpha_{i}, \alpha_{j}\right)+2 \leq \ell\left(C^{\prime}\right) \leq g$, a contradiction.

Therefore, $g\left(G^{\prime}\right)=g$ and $G^{\prime}$ is a $[1, r ; g]$-mixed graph. Thus,

$$
n[1, r ; g] \leq\left|V\left(G^{\prime}\right)\right|=|V(G)|-1<n[1, r ; g+1]
$$

and the result follows.
The case in which $C$ contains two consecutive arcs is analogous.
Case 1.2) Suppose that $r$ is odd. If $v_{1} u, u v_{2} \in E(C)$ (see Figure4), let $w=v_{r}, N(w)=\left\{v_{1}^{\prime}, \ldots, v_{r}^{\prime}\right\}$, where $v_{r}^{\prime}=u, N^{-}(w)=\left\{x^{\prime}\right\}$ and $N^{+}(w)=\left\{y^{\prime}\right\}$. Let $E^{\prime}=\left\{v_{2 i-1} v_{2 i}, v_{2 i-1}^{\prime} v_{2 i}^{\prime}: 1 \leq i \leq(r-1) / 2\right\} \cup\left\{\overrightarrow{x_{1} y_{1}}, \overrightarrow{x^{\prime} y^{\prime}}\right\}$.

Let $G^{\prime}=G-\{u, w\}+E^{\prime}$. Notice that $G^{\prime}$ contains a cycle of length $g$, therefore $g\left(G^{\prime}\right) \leq g$. Let $C^{\prime}$ be a cycle of $G^{\prime}$ such that $\ell\left(C^{\prime}\right)=g\left(G^{\prime}\right)$. By a similar analysis to the Case 1.1), it follows that $\ell\left(C^{\prime}\right) \geq d_{G-\{u, w\}}\left(\alpha_{i}, \alpha_{j}\right)+\left|E^{*}\left(C^{\prime}\right) \cap E^{\prime}\right| \geq g$, where $\alpha_{i}$ and $\alpha_{j}$ are vertices of the edges or arcs in $E^{*}\left(C^{\prime}\right) \cap E^{\prime}$.

Therefore, $G^{\prime}$ is a $[1, r ; g]$-mixed graph with $n[1, r ; g+1]-2$ vertices. Thus

$$
n[1, r ; g] \leq\left|V\left(G^{\prime}\right)\right|=|V(G)|-1<n[1, r ; g+1] .
$$



Figure 4: Operation in a $[1, r ; g+1]$-mixed cage with $r$ odd, in a cycle with two arcs consecutive.
The case in which $C$ contains two consecutive arcs is analogous.
Case 2) Suppose $z=2$. Let $N^{-}(u)=\left\{x_{1}, x_{2}\right\}$ and $N^{+}(u)=\left\{y_{1}, y_{2}\right\}$.
Case 2.1) Suppose that $r$ is even. If $v_{1} u, u v_{2} \in E(C)$, let $E^{\prime}=\left\{v_{2 i-1} v_{2 i}: 1 \leq i \leq r / 2\right\}$. Next, we define a set $A^{\prime}$ depending on the sets $N^{-}\left(y_{1}\right)$ and $N^{-}\left(y_{2}\right)$. If $x_{1} \in N^{-}\left(y_{1}\right)$ or $x_{2} \in N^{-}\left(y_{2}\right)$, then $A^{\prime}=\left\{\overrightarrow{x_{1} y_{2}}, \overrightarrow{x_{2} y_{1}}\right\}$. In other case, $A^{\prime}=\left\{\overrightarrow{x_{1} y_{1}}, \overrightarrow{x_{2} y_{2}}\right\}$. Let $G^{\prime}=G-u+E^{\prime}+A^{\prime}$ (see Figure 5). Proceeding as in Case 1.1), it follows that $g\left(G^{\prime}\right)=g$ and the result follows.

The case in which $\overrightarrow{x_{1} u}, \overrightarrow{u y_{1}} \in A(C)$ and $y_{2} \notin N^{+}\left(x_{2}\right)$ is proved in a similar way.


Figure 5: Operation in a $[2, r ; g+1]$-mixed cage with $r$ even, in a cycle with two edges consecutive and there is no the arc $\overrightarrow{x_{1} y_{1}}$ or $\overrightarrow{x_{2} y_{2}}$.

Suppose now that $\overrightarrow{x_{1} u}, \overrightarrow{u y_{1}} \in A(C)$ and $y_{2} \in N^{+}\left(x_{2}\right)$. If $y^{\prime} \in N^{+}\left(y_{2}\right) \cap V(C)$, let $E^{\prime}=\left\{v_{2 i-1} v_{2 i}: 1 \leq\right.$ $i \leq r / 2\} \cup\left\{\overrightarrow{x_{1} y_{2}}, \overrightarrow{x_{2} y_{1}}\right\}$. Let $G^{\prime}=G-u+E^{\prime}$ (see Figure 6). We claim that $g\left(G^{\prime}\right)=g$. Since $G^{\prime}$ contains the cycle $C-u+\overrightarrow{x_{1} y_{2}}+\overrightarrow{y_{2} y^{\prime}}$, it follows that $g\left(G^{\prime}\right) \leq g+1$. If $g\left(G^{\prime}\right)=g+1$, then $G^{\prime}$ is a $[2, r ; g+1]$-mixed graph with $n[2, r ; g+1]-1$ vertices, a contradiction. Therefore $g\left(G^{\prime}\right) \leq g$ and by a similar analysis to that in Case 1.1), it follows that $n[2, r ; g] \leq\left|V\left(G^{\prime}\right)\right|<|V(G)|$.

If $N^{+}\left(y_{2}\right) \cap V(C)=\emptyset$, let $N^{+}\left(y_{2}\right)=\left\{y_{3}, y_{4}\right\}, N\left(y_{2}\right)=\left\{w_{1}, \ldots, w_{r}\right\}$. Set $E^{\prime}=\left\{w_{1} w_{2}, w_{3} w_{4}, \ldots, w_{r-1} w_{r}\right\} \cup$ $\left\{\overrightarrow{u y_{3}}, \overrightarrow{x_{2} y_{4}}\right\}$. Let $G^{\prime}=G-y_{2}+E^{\prime}$, note that $G^{\prime}$ is a $\left[2, r, g^{\prime}\right]$-mixed graph, with one vertex less than $G$ and $g^{\prime}=g\left(G^{\prime}\right)<g+1$, since the cycle $C$ is contained in $G^{\prime}$. Let $C^{\prime}$ be a cycle such that $\ell\left(C^{\prime}\right)=g^{\prime}$, similarly to Case 1.1), we conclude that $\ell\left(C^{\prime}\right) \geq g$. Thus, $g^{\prime}=g$ and $G^{\prime}$ is a $[2, r ; g]$-mixed graph with $n[2, r ; g+1]-1$ vertices and $n[2, r ; g] \leq\left|V\left(G^{\prime}\right)\right|<n[2, r ; g+1]$.

Case 2.2) Suppose that $r$ is odd. Let $s=v_{r}, N(s)=\left\{v_{1}^{\prime}, \ldots, v_{r}^{\prime}\right\}$, where $v_{r}^{\prime}=u, N^{-}(s)=\left\{x_{1}^{\prime}, x_{2}^{\prime}\right\}$ and $N^{+}(s)=\left\{y_{1}^{\prime}, y_{2}^{\prime}\right\}$.

Suppose that $v_{1} u, u v_{2} \in E(C)$. Let $E^{\prime}=\left\{v_{2 i-1} v_{2 i}, v_{2 i-1}^{\prime} v_{2 i}^{\prime}: 1 \leq i \leq(r-1) / 2\right\} \cup A_{u} \cup A_{s}$, where $A_{u}$ and $A_{s}$ are defined depending on the sets $N^{-}\left(y_{i}\right)$ and $N^{-}\left(y_{i}^{\prime}\right)$ for $i \in\{1,2\}$. Since $d^{-}\left(y_{i}\right)=2$, it follows


Figure 6: Operation in a $[2, r ; g+1]$-mixed cage with $r$ even, in a cycle with two arcs consecutive and there is no the arc $\overrightarrow{x_{2} y_{2}}$.
that $\left|N^{-}\left(y_{i}\right) \cap\left\{x_{1}, x_{2}\right\}\right| \leq 1$. If either $x_{1} \in N^{-}\left(y_{1}\right)$ or $x_{2} \in N^{-}\left(y_{2}\right)$, then $A_{u}=\left\{\overrightarrow{x_{1} y_{2}}, \overrightarrow{x_{2} y_{1}}\right\}$. In other case set $A_{u}=\left\{\overrightarrow{x_{1} y_{1}}, \overrightarrow{x_{2} y_{2}}\right\}$. If either $x_{1}^{\prime} \in N^{-}\left(y_{1}^{\prime}\right)$ or $x_{2}^{\prime} \in N^{-}\left(y_{2}^{\prime}\right)$, then $A_{s}=\left\{\overrightarrow{x_{1}^{\prime} y_{2}^{\prime}}, \overrightarrow{x_{2}^{\prime} y_{1}^{\prime}}\right\}$. In other case $A_{s}=\left\{\overrightarrow{x_{1}^{\prime} y_{1}^{\prime}}, \overrightarrow{x_{1}^{\prime} y_{1}^{\prime}}\right\}$. Let $G^{\prime}=G-\{u, s\}+E^{\prime}$. Proceeding as in Case 1.1), it follows that $g\left(G^{\prime}\right)=g$, and the result follows.

The case in which $\overrightarrow{x_{1} u}, \overrightarrow{u y_{1}} \in A(C)$ and $y_{2} \notin N^{+}\left(x_{2}\right)$ is proved in a similar way.
Next, suppose that $\overrightarrow{x_{1} u}, \overrightarrow{u y_{1}} \in A(C)$, and $y_{2} \in N^{+}\left(x_{2}\right)$. If $y^{\prime} \in N^{+}\left(y_{2}\right) \cap V(C)$, let $E^{\prime}=\left\{v_{2 i-1} v_{2 i}, v_{2 i-1}^{\prime} v_{2 i}^{\prime}\right.$ : $1 \leq i \leq(r-1) / 2\} \cup\left\{\overrightarrow{x_{1} y_{2}}, \overrightarrow{x_{2} y_{1}}\right\} \cup A_{s}$. If either $x_{1}^{\prime} \in N^{-}\left(y_{1}^{\prime}\right)$ or $x_{2}^{\prime} \in N^{-}\left(y_{2}^{\prime}\right)$, then $A_{s}=\left\{\overrightarrow{x_{1}^{\prime} y_{2}^{\prime}}, \overrightarrow{x_{2}^{\prime} y_{1}^{\prime}}\right\}$. In other case, $A_{s}=\left\{\overrightarrow{x_{1}^{\prime} y_{1}^{\prime}}, \overrightarrow{x_{2}^{\prime} y_{2}^{\prime}}\right\}$. Let $G^{\prime}=G-\{u, s\}+E^{\prime}$ (see Figure 7). Since $G^{\prime}$ contains the cycle $C-u+\overrightarrow{x_{1} y_{2}}+\overrightarrow{y_{2} y^{\prime}}$, therefore $g\left(G^{\prime}\right) \leq g+1$. If $g\left(G^{\prime}\right)=g+1$, it follows that $G^{\prime}$ is a $[2, r ; g+1]$-mixed graph, a contradiction. Hence, $g\left(G^{\prime}\right) \leq g$ and proceeding as in Case 1.1), it follows that $n[2, r ; g] \leq\left|V\left(G^{\prime}\right)\right|<|V(G)|$.

If $N^{+}\left(y_{2}\right) \cap V(C)=\emptyset$, let $N^{+}\left(y_{2}\right)=\left\{y_{3}, y_{4}\right\}, N\left(y_{2}\right)=\left\{w_{1}, \ldots, w_{r}\right\}$, with $w_{r}=t$. Let $N(t)=$ $\left\{w_{1}^{\prime}, \ldots, w_{r}^{\prime}\right\}$, with $w_{r}^{\prime}=y_{2}, N^{-}(t)=\left\{x_{1}^{\prime}, x_{2}^{\prime}\right\}$ and $N^{+}(t)=\left\{y_{1}^{\prime}, y_{2}^{\prime}\right\}$. Let $E^{\prime}=\left\{w_{2 i-1} w_{2 i}, w_{2 i-1}^{\prime} w_{2 i}^{\prime}: 1 \leq\right.$ $i \leq(r-1) / 2\} \cup\left\{\overrightarrow{u y_{3}}, \overrightarrow{x_{2} y_{4}}\right\} \cup A_{t}$, where $A_{t}=\left\{\overrightarrow{x_{1}^{\prime} y_{2}^{\prime}}, \overrightarrow{x_{2}^{\prime} y_{1}^{\prime}}\right\}$ if either $x_{1}^{\prime} \in N^{-}\left(y_{1}^{\prime}\right)$ or $x_{2}^{\prime} \in N^{-}\left(y_{2}^{\prime}\right)$, and $A_{t}=\left\{\overrightarrow{x_{1}^{\prime} y_{1}^{\prime}}, \overrightarrow{x_{2}^{\prime} y_{2}^{\prime}}\right\}$ in any other case.

Let $G^{\prime}=G-\left\{y_{2}, t\right\}+E^{\prime}$. Since $C$ is contained in $G^{\prime}$, it follows that $g\left(G^{\prime}\right)<g+1$. Let $C^{\prime}$ a cycle such that $\ell\left(C^{\prime}\right)=g\left(G^{\prime}\right)$, proceeding as in Case 1.1), it can be concluded that $g\left(G^{\prime}\right)=g$. Hence, $G^{\prime}$ is a $[2, r ; g]$-mixed graph with $n[2, r ; g+1]-2$ vertices. Therefore $n[2, r ; g] \leq\left|V\left(G^{\prime}\right)\right|<n[2, r ; g+1]$, and the theorem is proved.

### 2.2 Connectivity of a mixed cage

In this subsection we give some results on the connectivity of a mixed cage. A mixed graph $G$ is strong if for every two vertices $u$ and $v$ of $G$ there exists a $u v$-path and a $v u$-path. Clearly, if $G$ is a $[z, r ; g]$-mixed cage, then the underlying graph of $G$ is connected.

Theorem 4. If $G$ is a $[z, r ; g]$-mixed cage, then $G$ is strongly connected.
Proof. Let $G$ be a $[z, r ; g]$-mixed cage. Suppose to the contrary that $G$ is not strong. Let $H_{1}, \ldots, H_{k}$ be the strong components of $G$. Note that there are no edges between the strong components of $G$. Let $H^{*}=\cup_{i=2}^{k} H_{i}$. Since the underlying graph of $G$ is connected, there is at least one arc between $H_{1}$ and $H^{*}$. Furthermore, all the arcs between $H_{1}$ and $H^{*}$ have the same direction. Suppose without lose of generality


Figure 7: Operation in a $[2, r ; g+1]$-mixed cage with $r$ odd, in a cycle with two consecutive arcs and there is the arc $\overrightarrow{x_{2} y_{2}}$.
that $\left[V\left(H_{1}\right), V\left(H^{*}\right)\right] \neq \emptyset$. The number of arcs of $G$ is

$$
|A(G)|=|V(G)| z=\left(\left|V\left(H_{1}\right)\right|+\left|V\left(H^{*}\right)\right|\right) z .
$$

On the other hand, for every vertex $v \in V\left(H_{1}\right)$ it follows that $d^{-}(v)=z$, and for every vertex $u \in$ $V\left(H^{*}\right), d^{+}(u)=z$. Hence, $\left|A\left(H_{1}\right)\right|=\left|V\left(H_{1}\right)\right| z$ and $\left|A\left(H^{*}\right)\right|=\left|V\left(H^{*}\right)\right| z$. Therefore, $|A(G)|=\left|A\left(H_{1}\right)\right|+$ $\left|A\left(H^{*}\right)\right|+\left|\left[V\left(H_{1}\right), V\left(H^{*}\right)\right]\right|=|V(H)| z+\left|V\left(H^{*}\right)\right| z+\left|\left[V\left(H_{1}\right), V\left(H^{*}\right)\right]\right|$, implying that $\left|\left[V\left(H_{1}\right), V\left(H^{*}\right)\right]\right|=0$, a contradiction.

Theorem 5. The underlying graph of $a[z, r ; g]$-mixed cage is 2-connected, for $z \in\{1,2\}$.
Proof. Let $G$ be a $[z, r ; g]$-mixed cage, $z \in\{1,2\}$. Suppose that there exists a vertex $v \in V(G)$ such that the underlying graph of $G-v$ is not connected. Let $H$ be a connected component of $G-v$ of minimum order. Observe that $|V(H)|<|V(G)| / 2$. Let $\overleftarrow{H}$ be the reverse graph of $H$ and let $u^{\prime}$ denote the corresponding vertex of $u$ in $\overleftarrow{H}$. We construct a new graph $G^{*}$ formed by the disjoint union of $H$ and $\overleftarrow{H}$, an edge set $E^{\prime}=\left\{u u^{\prime}: u \in V(H) \cap N(v), u^{\prime} \in V(\overleftarrow{H})\right\}$ and an arc set $A^{\prime}=\left\{\overrightarrow{u u^{\prime}}: u \in V(H) \cap N^{-}(v), u^{\prime} \in V(\overleftarrow{H})\right\} \cup\left\{\overrightarrow{u^{\prime} u}:\right.$ $\left.u \in V(H) \cap N^{+}(v), u^{\prime} \in V(\overleftarrow{H})\right\}$. Observe that $G^{*}$ is a $\left[z, r, g\left(G^{*}\right)\right]$-mixed graph. Let $C$ be a cycle of length $g\left(G^{*}\right)$. Since $\left|V\left(G^{*}\right)\right|=2|V(H)|<|V(G)|$, by Theorem $3, g\left(G^{*}\right)<g$. Hence $(E(C) \cup A(C)) \cap\left(E^{\prime} \cup A^{\prime}\right) \neq \emptyset$. Thus, there exists at least two vertices $u_{1}$ and $u_{2}$ of $H$ which are the endings of those edges or arcs belonging to $C$. Notice that $u_{1}$ and $u_{2}$ are at distance at least $g-2$ in $G-v$. Hence, $g\left(G^{*}\right) \geq 2(g-2)>g$. Therefore, $G^{*}$ is a $\left[z, r ; g^{*}\right]$-mixed cage with $g^{*} \geq g$. By Theorem 3, $n\left[z, r ; g^{\prime}\right] \leq\left|V\left(G^{*}\right)\right|<|V(G)|=n[z, r ; g]$, a contradiction.

## 3 Construction of mixed graphs

In this section some constructions of families of mixed graphs are presented.

### 3.1 Lower bounds

In this subsection we give a lower bound for $n[z, r ; g]$. Let $G$ be a mixed graph. Given a vertex $u$ of $G$, we define the projection of $u$ as $\vec{N}(u)=N^{+}(u) \cup N(u)$. Similarly, the injection of $u$ is the set $\overleftarrow{N}(u)=$ $N^{-}(u) \cup N(u)$.

Proposition 6. The order of $a[z, r ; g]$-mixed cage is at least $n_{0}(r, g)+2 z$.
Proof. Let $G$ be a $[z, r ; g]$-mixed cage. By deleting the arcs of $G$ we obtain an $\left(r, g^{\prime}\right)$-graph with $g^{\prime} \geq g$. Hence by the Moore bound and the monotonocity it follows that $|V(G)| \geq n_{0}(r, g)$. In addition, since every vertex of $G$ has $z$ ex-neighbors and $z$ in-neighbors, it follows that $\mid V(G) \geq n_{0}(r, g)+2 z$.

Next we improve the previous lower bound for some specific parameters.
Theorem 7. The order of $a[10,3 ; 5]$-mixed cage is at least 50.
Proof. Let $G$ be a $[10,3 ; 5]$-mixed cage. By Proposition 6, it follows that $|V(G)| \geq 30$. Let $G^{\prime}=G-A(G)$. Observe that $G^{\prime}$ is a $\left(3, g^{\prime}\right)$-graph with $g^{\prime} \geq 5$. Let $u \in V(G)$ and let $N_{2}(u)$ be the set of vertices of $G^{\prime}$ at distance at most 2 from $u$. Observe that $N^{+}(u) \cap N_{2}(u)=\emptyset$ and $N^{-}(u) \cap N_{2}(u)=\emptyset$.

Claim. There exists a vertex $v \in N^{+}(u)$ such that $\left|\vec{N}(v) \cap N^{+}(u)\right| \leq 3$.
Suppose that for every $v \in N^{+}(u),\left|\vec{N}(v) \cap N^{+}(u)\right| \geq 4$. Therefore, there exists a vertex $w \in N^{+}(u)$ such that $\left|N(w) \cap N^{+}(u)\right| \leq 2$. Otherwise the mixed graph induced by $N^{+}(u)$ would contains a cycle of length at most 4. Let $y \in N^{+}(u)$ and suppose that $\left|N(y) \cap N^{+}(u)\right|=2$. Let $x_{1}, x_{2} \in N(y) \cap N^{+}(u)$ and let $z \in N^{+}(y) \cap N^{+}(u)$. Since $\left|\vec{N}(z) \cap N^{+}(u)\right| \geq 4$ and $G$ has girth 5 , it follows that $x_{1}, x_{2} \notin \vec{N}(z)$. Let $Z=\vec{N}(z) \cap N^{+}(u)$ (see Figure 8).


Figure 8: Structure with two adjacent edges.

Notice that if we want maximize the number of edges and the arcs in $G[Z]$ (the subgraph induced graph by $Z$ ), there are only nine possibilities for $G[Z]$ (see Figure 9). If there is a vertex $w \in Z$ such that $|\vec{N}(w) \cap Z|=1$, then $w$ is incident with an edge of $G[Z]$ (see Figure 91. Hence, $w$ cannot be adjacent to $z$
with an edge, because the girth of $G$ is 5. Therefore $\left|\vec{N}(w) \cap\left(N^{+}(u) \backslash Z\right)\right| \geq 3$ and since $x_{1}, x_{2}, y \notin \vec{N}(w)$, a contradiction is obtained.


Figure 9: The nine configurations of $G[Z]$.
Similarly, in the possibilities of $G[Z]$ that have a vertex $w$ with $|\vec{N}(w) \cap Z|=0$, it follows that $\mid \vec{N}(w) \cap$ $\left(N^{+}(u) \backslash\{z\}\right) \mid \geq 3$. Since $x_{1}, x_{2}, y \notin \vec{N}(w)$, a contradiction is obtained. Consequently, there are no two incident edges in $N^{+}(u)$.

Let $x \in N(y) \cap N^{+}(u)$ and $z \in N^{+}(y) \cap N^{+}(u)$ (see Figure 10.). Observe that $\left|\vec{N}(z) \cap N^{+}(u)\right| \geq 4$ and


Figure 10: Structure with at least 11 vertices in $N^{+}(u)$.
$x, y \notin\left(\vec{N}(z) \cap N^{+}(u)\right)$. Let $Z=\vec{N}(z) \cap N^{+}(u)$. In this case we only have five possibilities for $G[Z]$ (see Figure 11.

Observe that in each one of the possible graphs of $G[Z]$ there is a vertex $w$ with either $|\vec{N}(w) \cap Z|=0$ or $|\vec{N}(w) \cap Z|=1$. Let $w \in Z$ and suppose that $|\vec{N}(w) \cap Z|=0$. Since $\left|\vec{N}(w) \cap N^{+}(u)\right| \geq 4$, it follows that $\left|N^{+}(u)\right| \geq 11$, a contradiction. If $|\vec{N}(w) \cap Z|=1$, then $|N(w) \cap Z|=1$. Moreover, since there are no two incident edges in $N^{+}(u)$, it follows that $w \in N^{+}(z)$ and there are at least three vertices $w_{1}, w_{2}, w_{3} \in\left(N^{+}(w) \cap\left(N^{+}(u) \backslash\{x, y, z\}\right)\right)$. Since $\left|\vec{N}\left(w_{1}\right) \cap N^{+}(u)\right| \geq 4$ and $\left|N^{+}(u)\right|=10$, it follows that


Figure 11: The five configurations of four vertices with the maximum number of edges and arcs preserving the five girth and without two adjacent edges.
$\left|\vec{N}\left(w_{1}\right) \cap(Z \cup\{y, z\})\right| \geq 1$. Therefore a cycle of length at most 4 is obtained. A contradiction.
Therefore, there is a vertex $w \in N^{+}(u)$ such that $\left|\vec{N}(w) \cap N^{+}(u)\right| \leq 3$. By a similar reasoning, there is a vertex $w^{*} \in N^{-}(u)$ such that $\left|\overleftarrow{N}\left(w^{*}\right) \cap N^{-}(u)\right| \leq 3$

Hence $\left(\vec{N}(w) \backslash N^{+}(u)\right) \cap\left(N^{-}(u) \cup N_{2}(u)\right)=\emptyset$ and $\left(\overleftarrow{N}\left(w^{*}\right) \backslash N^{-}(u)\right) \cap\left(N^{+}(u) \cup N_{2}(u)\right)=\emptyset$. Since $|\vec{N}(w)|=\left|\overleftarrow{N}\left(w^{*}\right)\right|=13$, it follows that $|V(G)| \geq 50$, and the result follows.

### 3.2 Upper bounds

### 3.2.1 A family of $[z, r ; 5]$-mixed graph

To construct this family of mixed graphs we use the incidence graph of a partial plane. A partial plane is defined as two finite sets $\mathscr{P}$ and $\mathscr{L}$ called points and lines, respectively, where $\mathscr{L}$ consists of subsets of $\mathscr{P}$, such that any line is incident with at least two points, and two points are incident with at most one line. The incidence graph of a partial plane is a bipartite graph with partite sets $\mathscr{P}$ and $\mathscr{L}$ where a point of $\mathscr{P}$ is adjacent to a line of $\mathscr{L}$ if they are incident. Observe that the incidence graph of a partial plane has even girth $g \geq 6$. In Remark 8 we describe a biaffine plane.

Remark 8. [8].
Let $\mathbb{F}_{q}$ be the finite field of order $q$.
(i) Let $\mathscr{L}=\mathbb{F}_{q} \times \mathbb{F}_{q}$ and $\mathscr{P}=\mathbb{F}_{q} \times \mathbb{F}_{q}$ denoting the elements of $\mathscr{L}$ and $\mathscr{P}$ using"brackets" and "parenthesis", respectively. The following set of $q^{2}$ lines define a biaffine plane:

$$
\begin{equation*}
[m, b]=\left\{(x, m x+b): x \in \mathbb{F}_{q}\right\} \text { for all } m, b \in \mathbb{F}_{q} \tag{1}
\end{equation*}
$$

(ii) The incidence graph of the biaffine plane is a bipartite graph $B_{q}=(\mathscr{P}, \mathscr{L})$ which is $q$-regular, has order $2 q^{2}$, diameter 4 and girth 6 , if $q \geq 3$; and girth 8 , if $q=2$.
(iii) The vertices mutually at distance 4 are the vertices of the sets $L_{m}=\left\{[m, b]: b \in \mathbb{F}_{q}\right\}$, and $P_{x}=$ $\left\{(x, y): y \in \mathbb{F}_{q}\right\}$ for all $x, m \in \mathbb{F}_{q}$.

Next, we describe two operations that we perform on the graph $B_{q}$ : reduction and amalgam.
The reduction operation refers to delete the last pairs of blocks $\left(P_{i}, L_{i}\right)$ from $B_{q}$. Let $\gamma \in\{1, \ldots, q-1\}$, define $B_{q}(\gamma)=B_{q}-\bigcup_{i=1}^{\gamma}\left(P_{q-i} \cup L_{q-i}\right)$.
Lemma 9. [3] Let $\gamma \in\{1, \ldots, q-1\}$. Then, the graph $B_{q}(\gamma)$ is $(q-\gamma)$-regular of order $2\left(q^{2}-q \gamma\right)$ and girth $g \geq 6$.

Now, we describe the amalgam operation. Let $\Gamma_{1}$ and $\Gamma_{2}$ be two graphs of the same order and with the same labels on their vertices. The amalgam of $\Gamma_{1}$ into $\Gamma_{2}$ is a graph obtained adding all the edges of $\Gamma_{1}$ to $\Gamma_{2}$.

We will show how we apply the operations of reduction and amalgam to the graph $B_{q}$ to build a family of $[z ; r ; 5]$-mixed graph.

Let $z \neq 2$ be a positive integer. Let $p$ be the smallest prime number such that $4 z+1 \leq p \leq 5 z$. Consider $\mathbb{F}_{p}=\mathbb{Z}_{p}$ and let $B_{p}$ be the incidence graph of the biaffine plane with this field. Let $\vec{C}_{p}(1, \ldots, z)$ the circulant digraph of order $p$. Recall that a circulant digraph over $\mathbb{Z}_{p}$, denoted by $\vec{C}_{p}(1, \ldots, z)$ is a digraph whose set of vertices are the elements of $\mathbb{Z}_{p}$ and the set of arcs in $\vec{C}_{p}(1, \ldots, z)$ are defined as $A\left(\vec{C}_{p}(1, \ldots, z)\right)=\{\overrightarrow{i j} \mid(j-i) \in\{1, \ldots, z\} \bmod p\}$. Let $\{1, \ldots, z\}$ be the weight or the Cayley color of the $\operatorname{arcs}$ on $\vec{C}_{p}(1, \ldots, z)$.

We define $B_{p}^{*}(\gamma)$ to be the amalgam of $\vec{C}_{p}(1, \ldots, z)$ into $P_{i}$ and $L_{i}$ for $i \in\{0, \ldots, p-\gamma-1\}$ and $\gamma \in\{1, \ldots, p-2\}$. To simplify notation, we assume that the labelling of $\vec{C}_{p}(1, \ldots, z)$ corresponds to the second coordinate of $P_{i}$ and $L_{i}$ for $i \in\{1, \ldots, p-\gamma-1\}$.

To prove Theorem 10, we use a result due to Dusart 9. For any integer $z \geq 3275$, there always exists a prime number between $n$ and $\left(1+1 /\left(2 \ln ^{2} n\right)\right) n$. Since $\left(1+1 /\left(2 \ln ^{2}(4 z+1)\right)\right)(4 z+1)<5 z$, we ensure that for any integer $z \geq 3275$, there always exists a prime number between $4 z+1$ and $5 z$. And, it is not difficult to obtain, using simple computer calculations, the same result for $1 \leq z \leq 3275$ and $z \neq 2$.

Theorem 10. Let $p$ be the smallest prime number such that $4 z+1 \leq p \leq 5 z$, for every positive integer $z \neq 2$. Then $n[z, r ; 5] \leq 2 p r$, for $r \in\{1, \ldots, p\}$.

Proof. Let $p$ be the smallest prime number such that $4 z+1 \leq \underline{p} \leq 5 z$, for every positive integer $z$ other than 2. Let $\vec{C}_{p}(1, \ldots, z)$ be the circulant digraph. Observe that $\vec{C}_{p}(1, \ldots, z)$ has girth 5 . Let $B_{p}^{*}(\gamma)$ be the amalgam of $\vec{C}_{p}(1, \ldots, z)$ into $P_{i}$ and $L_{i}$ for $i \in\{0, \ldots, p-\gamma-1\}$ and $\gamma \in\{1, \ldots, p-2\}$. Let $r=p-\gamma$. Notice that $\left|B_{p}^{*}(\gamma)\right|=2 p(p-\gamma)=2 p r$, also each vertex $v \in V\left(B_{p}^{*}(\gamma)\right)$ is $r$-regular in edges and $z$-regular in arcs. Let $C$ be a shortest cycle in $B_{p}^{*}(\gamma)$. Suppose by contradiction, that $|V(C)| \leq 4$. Therefore, $C=(w, x, y, w)$ or $C=(v, w, x, y, v)$. Notice that $C$ cannot be completely contained in $\vec{C}_{p}(1, \ldots, z)$ or in $B_{p}(\gamma)$. With out loose of generality suppose that $w, x \in P_{i}$ and $y \in L_{m}$ for some $i, m \in\{0,1, \ldots, r\}$, that is, $w=(i, a)$, $x=(i, b)$ and $y=[m, k]$. Since $\overrightarrow{w x} \in A\left(B_{p}^{*}(\gamma)\left[P_{i}\right]\right)$, then $b=a+s$, for some $s \in\{1, \ldots, z\}$. Since the edges between $P_{i}$ and $L_{m}$ induces a matching, then $w y \notin E\left(B_{p}(\gamma)\right)$, and hence $w y \notin E\left(B_{p}^{*}(\gamma)\right)$. Thus $|V(C)|>3$, and we can assume $|V(C)|=4$ and $C=(v, w, x, y, v)$. By the same argument, $v \notin P_{i}$. Since there are no edges between $P_{i}$ and $P_{j}$ in $B_{p}^{*}(\gamma)$, for $j \in\{0,1, \ldots, r\} \backslash\{i\}$, neither between $L_{m}$ and $L_{n}$ in $B_{p}^{*}(\gamma)$, for $n \in\{0,1, \ldots, r\} \backslash\{m\}$. Thus $v=[m, l] \in L_{m}$ and $\overrightarrow{y v} \in A\left(B_{p}^{*}(\gamma)\left[L_{m}\right]\right)$, then $l=k+t$, for some $t \in\{1, \ldots, z\}$. If $x y \in E\left(B_{p}\right)$, then $x=(i, b)=(i, a+s)$ and $y=[m, k]$. Hence, $a+s=m i+k$, implying $y=[m, a+s-m i]$. Since $v w \in E\left(B_{p}\right), v=[m, l]$ and $w=(i, a)$ it follows that $a=m i+l$, that is, $v=[m, a-m i]$. By definition of $\vec{C}_{q}(1, \ldots, z)$, we have that $\overrightarrow{v y} \in A\left(\vec{C}_{p}(1, \ldots, z)\right)$ instead of $\vec{y} \vec{v}$, a contradiction. Hence $g\left(B_{p}^{*}(\gamma)\right)=5$ and $B_{p}^{*}(\gamma)$ is a $[z, r ; 5]$-mixed graph of order $2 p r$. Therefore, $n[z, r ; 5] \leq 2 p r$.

If $r=p$, then simply amalgam $\vec{C}_{p}(1, \ldots, z)$ in each $P_{i}$ and $L_{i}$ of $B_{p}$, and by a similar analysis we verify that $g\left(B_{p}^{*}\right)=5$, it follows that $n[z, r ; 5] \leq 2 p r$.

In Figure 12 is depicted an example of a $[3,13 ; 5]$-mixed graph that is an amalgam of the circulant digraph $\vec{C}_{13}(1,2,3)$ in $B_{13}$.

### 3.2.2 Other bounds for different girth

In this subsection we present an upper bound for $n[z, r ; g]$.
Theorem 11. Let $r \geq 2$ and $g \geq 3$ be integers. Then $n\left[z^{\prime}, r ; g\right] \leq g n_{0}(r, g)$, for $1 \leq z^{\prime} \leq n_{0}(r, g)$.
Proof. We present a construction of a $\left[z^{\prime}, r ; g\right]$-mixed graph of order $g n_{0}(r, g)$, for every $z^{\prime} \in\left\{1,2, \ldots, n_{0}\right\}$. Let $H$ be an $(r, g)$-cage. Suppose that $V(H)=\left\{1,2, \ldots, n_{0}(r, g)\right\}$. Let $H_{i}$ be a copy of $H$, for $i \in\{0,1, \ldots, g-1\}$, with $V\left(H_{i}\right)=\left\{1_{i}, 2_{i}, \ldots, n_{0, i}\right\}$. Let $G=\bigcup_{i=0}^{g-1} H_{i}$. Notice that $G$ is a disconnected $r$-regular graph with girth $g$ and order $g n_{0}(r, g)$. Let $B\left(H_{i}, H_{i+1}\right)$ be the complete bipartite directed graph that is obtained by adding all the arcs from $V\left(H_{i}\right)$ to $V\left(H_{i+1}\right)$, for $i \in\{0, \ldots, g-1\}(\bmod g)$. Let $\mathcal{A}_{i}=\left\{A_{1}, A_{2}, \ldots, A_{n_{0}(r, g)}\right\}$


Figure 12: A $[3,13 ; 5]$-mixed graph.
be a 1-factor (oriented) of $B\left(H_{i}, H_{i+1}\right)$. Let $\overrightarrow{\mathcal{A}_{i}}(j)=\bigcup_{i=1}^{j} \mathcal{A}_{i}$ and let $G^{*}=G+\bigcup_{i=0}^{g-1} \overrightarrow{\mathcal{A}_{i}}\left(z^{\prime}\right)$. Observe that we can always get that $\mathcal{A}_{i} \cap \mathcal{A}_{j}=\emptyset$, it follows that $G^{*}$ is a $\left[z^{\prime}, r ; g\right]$-mixed graph.

In the following we will prove that $G^{*}$ has girth $g$. Suppose that $C$ is a cycle such that $|V(C)|<g$, hence it cannot contain edges only. Neither can it consist only of arrows, by the cycles formed of arcs have length a multiple of $g$. Therefore $C$ consists of edges and arcs, that is, if it contains vertices of the copy $i$, then it contains at least one vertex of the copy $i+1$, and according to the direction of the arcs, the minimum distance of the vertices of the copy $i+1$ to any of the copy $i$ is $g-1$, which is a contradiction. Therefore $G^{*}$ has a girth $g$ and is a $\left[z^{\prime}, r ; g\right]$-mixed graph with $g n_{0}$ vertices, that is, $n\left[z^{\prime}, r ; g\right] \leq g n_{0}(r, g)$.

Corollary 12. There exists a $[10,3 ; 5]$-mixed cage of order 50 .
Proof. Let $G$ be a $[10,3 ; 5]$-mixed cage. By Theorem 7 and Theorem 11 , it follows that $|V(G)|=50$. The mixed graph depicted in Figure 13 is a $[10,3 ; 5]$-mixed cage.


Figure 13: A $[10,3 ; 5]$-mixed cage of order 50.

## 4 Future work

The problem of find a mixed cage and study their properties is very recently. As a suggestion to continue with the topic we propose two problems:

1. The study of the monotonocity for $[z, r ; g]$-mixed cages with $z \geq 3$.
2. Find better lower upper bounds for $n[z, r ; g]$, specially for $g=5$ and also find new constructions of $[z, r ; g]$-mixed graphs with few vertices for any $g \geq 5$. A natural suggestion should be study the case for $g=6$.

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