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ACHROMATIC NUMBERS FOR CIRCULANT GRAPHS AND DIGRAPHS

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Abstract

In this paper, we determine the achromatic and diachromatic numbers of some circulant graphs and digraphs each one with two lengths and give bounds for other circulant graphs and digraphs with two lengths. In particular, for the achromatic number we state that $\alpha(C_{16q^2+20q+7}(1,2)) = 8q+5$, and for the diachromatic number we state that $dac(\overrightarrow{C}_{32q^2+24q+5}(1,2)) = 8q+3$. In general, we give the lower bounds $\alpha(C_{4q^2+aq+1}(1,a)) \geq 4q+1$ and $dac(\overrightarrow{C}_{8q^2+2(a+4)q+a+3}(1,a)) \geq 4q+3$ when a is a non quadratic residue of $\mathbb{Z}_{4q+1}$ for graphs and $\mathbb{Z}_{4q+3}$ for digraphs, and the equality is attained, in both cases, for $a = 3$.

Finally, we determine the achromatic index for circulant graphs of $q^2 + q + 1$ vertices when the projective cyclic plane of odd order q exists.

Keywords: circulant graphs, complete colorings, achromatic number, achromatic index.

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1. Introduction

A *complete* k-vertex-coloring of a graph G is a vertex-coloring of G using k colors such that for every pair of colors there are at least two incident vertices in G colored with this pair of colors. The *chromatic* $\chi(G)$ and *achromatic* $\alpha(G)$ numbers of G are the smallest and the largest number of colors in a complete proper k-vertex-coloring of G. Therefore

$$\chi(G) \leq \alpha(G). \tag{1}$$

The concept of the achromatic number has been intensely studied in graphs since it was introduced by Harary, Hedetniemi and Prins [13] in 1967. The achromatic index $\alpha_1(G)$ is defined similarly to the achromatic number $\alpha(G)$ but with edges instead of vertices; for more references of results related to these parameter see for instance [2–5, 9, 10, 14, 24].

The chromatic number and achromatic number have been generalized for digraphs by several authors [12,21]. In particular, the dichromatic number, defined by Neumann-Lara [20], and the diachromatic number, defined by the authors [1], generalize the concepts of chromatic and achromatic numbers, respectively. An *acyclic* k-vertex-coloring of a digraph D is a k-vertex coloring, such that D has no monochromatic cycles and a *complete* k-vertex-coloring of a digraph D is a vertex coloring using k colors such that for every ordered pair (i, j) of different colors, there is at least one arc (u, v) such that u has color i and v has color j. The dichromatic number $dc(D)$ and diachromatic number $dac(D)$ of D are the smallest and the largest number of colors in a complete acyclic k-vertex-coloring of D. Therefore

$$dc(D) \leq dac(D). \tag{2}$$

Given a set $J \subseteq \{1, \ldots, n-1\}$, the *circulant digraph* $\overrightarrow{C}_n(J)$ is defined as the digraph with vertex set equal to $\mathbb{Z}_n$ and $A(\overrightarrow{C}_n(J)) = \{ij : j - i \equiv s \pmod{n}, s \in J\}$, we call J the set of *lengths* of $\overrightarrow{C}_n(J)$. Moreover, to obtain a *circulant graph* $C_n(J)$ we define the edges of $C_n(J)$, where in this case $J \subseteq \left\{1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$, as $E(C_n(J)) = \{ij : |j - i| \equiv s \pmod{n}, s \in J\}$.

In [11] it was obtained asymptotically results of the achromatic and harmonious numbers of circulant graphs. In [14] it was determined that the achromatic number of the cycle C_n is equal to its achromatic index, that is, $\alpha(C_n) = \alpha_1(C_n) = \max\left\{k : k \left\lfloor \frac{k}{2} \right\rfloor \leq n\right\} - s(n)$, where $s(n)$ is the number of positive integer solutions to $n = 2x^2 + x + 1$. On the other hand, in the collection of papers [4, 6, 7, 15, 17, 18, 22] the achromatic index of K_n was determined for $n \leq 14$, when $n = p^2+p+1$ for p an odd prime power and $n = q^2+2q+1-a$ for q a power of two and $a \in \{0, 1, 2\}$. For the no-proper version of these results see [2,3,16,23].

In this paper, we determine the achromatic and diachromatic numbers of some circulant graphs and digraphs each one with two lengths and we give bounds for other circulant graphs and digraphs with two lengths. In particular, for the achromatic number we state that $\alpha(C_{16q^2+20q+7}(1,2)) = 8q+5$, whenever $8q+5$ is a prime, and for the diachromatic number we state that $dac(\overrightarrow{C}_{32q^2+24q+5}(1,2)) = 8q+3$, whenever $8q+3$ is a prime. In general, we give the lower bounds $\alpha(C_{4q^2+aq+1}(1,a)) \geq 4q+1$ and $dac(\overrightarrow{C}_{8q^2+2(a+4)q+a+3}(1,a)) \geq 4q+3$ when a is a non quadratic residue of $\mathbb{Z}_{4q+1}$ for graphs and $\mathbb{Z}_{4q+3}$ for digraphs, and the equality is attained, in both cases, for $a=3$. In the last section, we determine the achromatic index for circulant graphs of q^2+q+1 vertices when the projective cyclic plane of odd order q exists.

2. Complete Colorings on Circulant Graphs and Digraphs

In this section we consider circulant graphs and digraphs of order prime p. We use simple upper bounds for the achromatic and diachromatic number and some properties of quadratic and non-quadratic residues.

An upper bound of the achromatic number of a graph G with size m is

$$\alpha(G) \leq \left\lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + 2m} \right\rfloor . \tag{3}$$

The authors [4] determined the following upper bound of the diachromatic number of a digraph D with size m

$$dac(G) \leq \left\lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + m} \right\rfloor . \tag{4}$$

First to continue we give the definition and some properties about quadratic residuos and non quadratic residuos of odd primes that we will use in order to prove our results (see [8]).

Definition. Let p be an odd prime and $\gcd(a,p)=1$. If the quadratic congruence $x^2 \equiv a \pmod{p}$ has a solution, then a is said to be a *quadratic residue of* p, denoted here by QR. Otherwise, a is called a *quadratic nonresidue of* p, and denoted by NQR.

In the following Remark we list some properties, about quadratic and non quadratic residues, depending if p is an odd prime congruent with either 1 or 3 modulo 4.

Remark 1. Let p be an odd prime and consider $\mathbb{Z}_p$. If $p \equiv 1 \pmod{4}$, then we have the following properties.

1. If $i \in QR$ ($i \in NQR$, respectivley), then $-i \in QR$ ($-i \in NQR$, respectively).
2. Let $i \in QR$. If $j \in QR$ ($j \in NQR$, respectively), then $ij \in QR$ ($ij \in NQR$, respectively).
3. The integer $2 \in NQR$ if and only if $p = 8q + 5$.

If $p \equiv 3 \pmod 4$, then we have the following.

1. If $i \in QR$ ($i \in NQR$, respectively), then $-i \in NQR$ ($-i \in QR$, respectively).
2. The integer $2 \in NQR$ if and only if $p = 8q + 3$.

In both cases $|QR| = \left\lfloor \frac{p-1}{2} \right\rfloor$.

2.1. Achromatic number on circulant graphs

Let p be a an odd prime power such that $p = 4q + 1$ and let QR be the set of quadratic residues and NQR the set of non residues quadratics in $\mathbb{Z}_p$.

Theorem 2. *Let $4q + 1$ be an odd prime number and let $a \in NQR$. Then $\alpha(C_{4q^2+aq+1}(1, a)) \geq 4q + 1$.*

Proof. Before we start the proof, we can note that even though Remark 1 states that for any p odd prime $|QR| = \left\lfloor \frac{p-1}{2} \right\rfloor$, it also states that for $p = 4q + 1$, if $a \in QR$, then $-a \in QR$, and as we construct a circulant graph we only consider one of these values in QR.

To prove the theorem we give a complete and proper coloring of C_{4q^2+aq+1} $(1, a)$ with $4q + 1$ colors, that is, K_{4q+1} is a homeomorphic image of the graph $C_{4q^2+aq+1}(1, a)$. Let K_{4q+1} be the complete graph and consider the quadratic residue $QR_{4q+1} = \{1, r_2, \ldots, r_q\}$ in $\mathbb{Z}_{4q+1}$.

We define the following color sequences, where $2 \leq i \leq q$.

$$
\begin{aligned}
R_1 &= (0, 1, 2, \ldots, 4q, 0, 1, \ldots, a-3, a-2),\\
R_2 &= (a-1, a-1+r_2, a-1+2r_2, \ldots, a-1+(4q)r_2, a-1,\\
&\quad\ a-1+r_2, a-1+2r_2, \ldots, a-1+(a-2)r_2),\\
R_3 &= (a-1+(a-1)r_2, a-1+(a-1)r_2+r_3, a-1+(a-1)r_2+2r_3, \ldots,\\
&\quad\ a-1+(a-1)r_2+(4q)r_3,\\
&\quad\ a-1+(a-1)r_2, a-1+(a-1)r_2+r_3, \ldots, a-1+(a-1)r_2+(a-2)r_3),\\
R_i &= (a-1+(a-1)r_2+(a-1)r_3+\cdots+(a-1)r_{i-1},\\
&\quad\ a-1+(a-1)r_2+(a-1)r_3+\cdots+(a-1)r_{i-1}+r_i,\\
&\quad\ a-1+(a-1)r_2+(a-1)r_3+\cdots+(a-1)r_{i-1}+2r_i, \ldots,\\
&\quad\ a-1+(a-1)r_2+(a-1)r_3+\cdots+(a-1)r_{i-1}+(4q)r_i,\\
&\quad\ a-1+(a-1)r_2+(a-1)r_3+\cdots+(a-1)r_{i-1},\\
&\quad\ a-1+(a-1)r_2+(a-1)r_3+\cdots+(a-1)r_{i-1}+r_i,
\end{aligned}
$$

$$\begin{aligned}
& a-1+(a-1)r_2+(a-1)r_3+\cdots+(a-1)r_{i-1}+2r_i,\ldots,\\
& a-1+(a-1)r_2+(a-1)r_3+\cdots+(a-1)r_{i-1}+(a-2)r_i),\\
R_q = {} & (a-1+(a-1)r_2+(a-1)r_3+\cdots+(a-1)r_{q-1},\\
& a-1+(a-1)r_2+(a-1)r_3+\cdots+(a-1)r_{q-1}+r_q,\\
& a-1+(a-1)r_2+(a-1)r_3+\cdots+(a-1)r_{q-1}+2r_q,\ldots\\
& a-1+(a-1)r_2+(a-1)r_3+\cdots+(a-1)r_{q-1}+(4q)r_q\\
& a-1+(a-1)r_2+(a-1)r_3+\cdots+(a-1)r_{q-1},\\
& a-1+(a-1)r_2+(a-1)r_3+\cdots+(a-1)r_{q-1}+r_q,\\
& a-1+(a-1)r_2+(a-1)r_3+\cdots+(a-1)r_{q-1}+2r_q,\ldots\\
& a-1+(a-1)r_2+(a-1)r_3+\cdots+(a-1)r_{q-1}+(a-2)r_q),\\
R_{q+1} = {} & (a-1+(a-1)r_2+(a-1)r_3+\cdots+(a-1)r_q).
\end{aligned}$$

For $1 \le i \le q$, we concatenate R_i with R_{i+1} and finally, we concatenate R_{q+1} with R_1, obtaining a $4q+1$-coloring of the cycle C_{4q^2+aq+1}. We add the edges between vertices of distance a on the cycle C_{4q^2+aq+1} obtaining the graph $C_{4q^2+aq+1}(1,a)$. Observe that, in addition to the edges between vertices of distance a in each R_i, for $1 \le i \le q$, between R_i and R_{i+1}, we have the edge between the $(4q+1)$-th vertex of R_i and the first vertex of R_{i+1},

$$\begin{aligned}
&\{a-1+(a-1)r_2+(a-1)r_3+\cdots+(a-1)r_{i-1}+(4q)r_i,\\
&\ a-1+(a-1)r_2+(a-1)r_3+\cdots+(a-1)r_i\}.
\end{aligned}$$

For instance, between R_1 and R_2 we add the edges $(a-2,a-1)$, $(4q,a-1)$. Finally, between R_{q+1} and R_1, we have the edge between the unique vertex of R_{q+1} and the a-th vertex of R_1,

$$\{a-1+(a-1)r_2+(a-1)r_3+\cdots+(a-1)r_q, a-1\}.$$

Summarizing, the concatenation $(R_{q+1}R_1\cdots R_qR_{q+1})$ in cyclic order contains all the edges $\{l,k\}$ for every pair $\{l,k\} \in Z_{4q+1}$, such that in R_i are the edges $\{j,j+r_i\}$ and $\{j,j+ar_i\}$ with $r_i \in QR_{4q+1}$ and $ar_i \in NQR_{4q+1}$ less the edges $\{a-1+(a-1)r_2+\cdots+(a-1)r_{i-1}+4qr_i, a-1+(a-1)r_2+\cdots+(a-1)r_i\}$.

For $1 \le i \le q$, each sequence R_i has order $4q+a$ and R_{q+1} has only one element, then the order of circulant graph is equal to $4q^2+aq+1$ and we can embed $C_{4q^2+aq+1}(1,a)$ in K_{4q+1}, and

$$4q+1 \le \alpha(C_{4q^2+aq+1}(1,a)). \qquad \blacksquare$$

By Remark 1, $2 \in NQR$ if and only if $p = 8q+5$, applying Equation 3 for $a = 2$, we have the following result.

Corollary 3. *Let $8q+5$ be a prime number. Then $\alpha(C_{16q^2+20q+7}(1,2)) = 8q+5$.*

Moreover, when $a = 3$ we have also the following.

Corollary 4. *Let* $4q+1$ *be a prime number. If* $3 \in NQR$, *then* $\alpha(C_{4q^2+aq+1}(1,3)) = 4q+1$.

Proof. If $a = 3$, $C_{4q^2+3q+1}(1,3)$ has $8q^2+6q+2$ edges and K_{4q+2} has $8q^2+6q+1$ edges, by Equation 3, it follows that $\alpha(C_{4q^2+3q+1}(1,3)) \leq 4q+2$. Suppose that $\alpha(C_{4q^2+3q+1}(1,3)) = 4q+2$. Observe that there is only one pair of colors repeated in the coloring of the graph $C_{4q^2+3q+1}(1,3)$. Consider a homomorphism $\varphi : V(C_{4q^2+3q+1}(1,3)) \to V(K_{4q+2})$. Let us consider the multigraph induced by the homomorphism φ. Since K_{4q+2} is $(4q+1)$-regular and $C_{4q^2+3q+1}(1,3)$ is 4-regular, then each vertex of the multigraph has degree at least $4q+4$. Hence, there are at least $3(4q+2)/2$ set of colors repeated (or multiedges), a contradiction. Thus, $\alpha(C_{4q^2+3q+1}(1,3)) = 4q+1$. ■

For instance, applying Corollary 3, for $q = 1$ we have that $8q+5 = 13$, and the quadratic residues in $\mathbb{Z}_{13}$ are $QR = \{1,3,4\}$, and by the Corollary 3, $\alpha(C_{43}(1,2)) = 13$. In this case the coloring is defined by the following sequences.

$$\begin{aligned}
R_1 &= (0,1,2,3,4,5,6,7,8,9,10,11,12,0),\\
R_2 &= (1,4,7,10,0,3,6,9,12,2,5,8,11,1),\\
R_3 &= (4,8,12,3,7,11,2,6,10,1,5,9,0,4),\\
R_4 &= (8).
\end{aligned}$$

2.2. Diachromatic number on circulant digraphs

Let $p = 4q+3$ be a prime. In this case $\overrightarrow{C}_p(J)$ is a circulant digraph for J, recall that by Remark 1 if $i \in QR$, then $-i \in NQR$.

Theorem 5. *Let* $4q+3$ *be a prime number and let* $a \in NQR$. *Then* $dac(\overrightarrow{C}_{8q^2+2(a+4)q+a+3}(1,a)) \geq 4q+3$. *Moreover, for* $a = 3$ *the equality holds.*

Proof. We proceed as in the proof of Theorem 2, considering the $2q+1$ quadratic residues and thus $2q+1$ sequences $R_1, R_2, \ldots, R_{2q+1}$ each one of order $4q+a+2$ and R_{2q+2} of order 1.

For the inequalities, when $a = 3$ proceed as in Corollary 4 considering the complete digraph K_{4q+4} with $(4q+4)(4q+3)$ arcs and $(4q+3)$ (in/out)-regular. In this case the size of K_{4q+4} equals the size of $\overrightarrow{C}_{8q^2+14q+6}(1,3)$ and the (in/out)-degree of each vertex of the multidigraph is congruent with 0 modulo 4. Since the complete digraph K_{4q+4} is $(4q+3)$ (in/out)-regular, for any pair of colors i and j there are $4q+3$ arcs from i to j, but that is impossible because $\overrightarrow{C}_{8q^2+14q+6}(1,3)$ has unique edges between many pair of colors. ■

Moreover, applying Remark 1 to Theorem 5 and the Equation 4 we have also the following result.

Corollary 6. *Let $8q+3$ be a prime number. Then* $dac(\overrightarrow{C}_{32q^2+24q+5}(1,2)) = 8q+3$.

For instance, if $q=1$, then $8q+3=11$, the quadratic residues in $\mathbb{Z}_{11}$ are $QR=\{1,3,4,5,9\}$, and by Corollary 6, $dac(\overrightarrow{C}_{61}(1,2))=11$. In this case the coloring is defined by the following sequences.

$$\begin{aligned}
R_1 &= (0,1,2,3,4,5,6,7,8,9,10,0),\\
R_2 &= (1,4,7,10,2,5,8,0,3,6,9,1),\\
R_3 &= (4,8,1,5,9,2,6,10,3,7,0,4),\\
R_4 &= (8,2,7,1,6,0,5,10,4,9,3,8),\\
R_5 &= (2,0,9,7,5,3,1,10,8,6,4,2),\\
R_6 &= (0).
\end{aligned}$$

3. Achromatic Index of Circulant Graphs

In this section, we endeavor to determine the achromatic index of some circulant graphs. In order to obtain upper bounds, we analyze the behavior of some functions. While to obtain lower bounds, we exhibit a proper and complete edge-coloring of $C_n(J)$ using some definitions and remarks about projective planes.

A *projective plane* consists of a set of n points, a set of lines, and an incidence relation between points and lines having the following properties.

- Given any two different points there is exactly one line incident to both of them.
- Given any two different lines there is exactly one point incident to both of them.
- There are four points, such that no line is incident to more than two of them.

For some number q, such a plane has $n=q^2+q+1$ points and n lines; each line contains $q+1$ points and each point belongs to $q+1$ lines. The number q is called the *order* of the projective plane. If a projective plane of order q exists, then it is denoted by Π_q. When q is a prime power, a projective plane of order q does exist which is denoted by $PG(2,q)$ owing to the fact that it arising from the finite field of order q and it is called *the algebraic projective plane*.

Let $\mathbb{P}$ be the set of points of Π_q and let $\mathbb{L}=\{L_0,\ldots,L_{n-1}\}$ be the set of lines of Π_q. The complete graph K_n can be seen to have the vertex set $\mathbb{P}$, and, for $i\in\{0,\ldots,n-1\}$, the line L_i can be interpreted as the subgraph K_{q+1} of K_n induced by the points of L_j; we denote this graph by L_i. From the properties of Π_q it follows that if $i,\ j\in\{0,\ldots,n-1\}$, $i\neq j$, then $|V(L_i)\cap V(L_j)|=1$. Moreover, $\{E(l_0),\ldots,E(l_{n-1})\}$ is a partition of $E(K_n)$.

Now, we recall the concept of difference sets of a $\mathbb{Z}_n$ group. A subset $D=\{d_0,d_1,\ldots,d_q\}$ of $\mathbb{Z}_n$ is a *difference set* if for each $g\in\mathbb{Z}_n$, $g\neq 0$, there is a unique pair of different elements $d_i, d_j \in D$ such that $g = d_i - d_j$. Therefore, if $D \subset \mathbb{Z}_n$ is a difference set and $i \in \mathbb{Z}_n$ is an arbitrary element, then both $-D := \{-d_0, -d_1, \ldots, -d_q\}$ and $D + i := \{d_0 + i, d_1 + i, \ldots, d_q + i\}$ are difference sets. See Table 1 for some examples.

$\mathbb{Z}_n$	Difference set	$\mathbb{Z}_n$	Difference set
$\mathbb{Z}_{13}$	1,2,5,7	$\mathbb{Z}_{91}$	1,2,4,10,28,50,57,62,78,82
$\mathbb{Z}_{31}$	1,2,4,9,13,19	$\mathbb{Z}_{133}$	1,2,4,13,21,35,39,82,89,95,105,110
$\mathbb{Z}_{57}$	1,2,4,14,33,37,44,53	$\mathbb{Z}_{183}$	1,2,4,17,24,29,43,77,83,87,120,138,155,176

Table 1. Some known different sets for some $\mathbb{Z}_n$.

Next, we recall the notion of cyclic planes using the polygon model. Let $q > 1$ be an integer, and $n = q^2 + q + 1$. If the group $\mathbb{Z}_n$ contains a difference set $D = \{d_0, d_1, \ldots, d_q\}$, then there exists a projective plane called *cyclic projective plane of order* q, defined as follows: the points are the elements of $\mathbb{Z}_n$ that is the set of integers $\{0, 1, \ldots, n-1\}$, and the lines are the sets $\{L_i\}_{i=0}^{n-1}$, where $L_i = D + i$. Throughout the paper when we deal with elements of $\mathbb{Z}_n$ all sums are taken modulo n.

The class of cyclic projective planes is wider than the class of algebraic projective planes, but each known finite cyclic plane is isomorphic to $PG(2, q)$ for a suitable q. The following representation of the cyclic planes comes from Kárteszi [19], and it is useful to illustrate our proofs. Consider the numbering of the vertices of a regular n-gon with the elements of $\mathbb{Z}_n$ in clockwise order. Note that the subpolygon with $q + 1$ vertices induced by a difference set D has the property that all the chords obtained by joining pairs of their points have different lengths, and it represents the line L_0 of Π_q. Moreover the line L_i is obtained by rotating L_0 around the center by angle $2\pi\frac{i}{n}$ for $i \in \{1, \ldots, n-1\}$, see Figure 1.

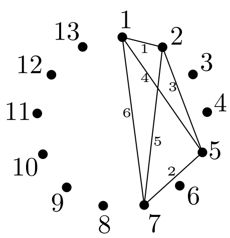

Figure 1. The line L_0 of the cyclic model of Π_3 as a polygon of 13 vertices, i.e., the line L_0 of the circulant graph $C_{13}(1, 2, 3, 4, 5, 6)$.

Finally, we recall that any complete graph of even order $q + 1$ admits a 1-factorization with q perfect matchings and we define the following definitions in order to simplify the proofs.

Definition. Let $q+1$ be an even integer, $\{F_1,\ldots,F_t\}$ a set of perfect matching pairwise edge-disjoint of K_{q+1} for some $t \in \{1,\ldots,q\}$ and $W = \bigcup_{j=1}^{t} F_j$. An edge-coloring $\varsigma\colon E(W) \to \mathcal{C}$, for $\mathcal{C} = \{c^1,\ldots,c^t\}$ a set of t colors, will say to be of *Type* M_t if for every $j \in \{1,2,\ldots,t\}$ the set $\{xy \in E(W)\colon \varsigma(xy) = c^j\}$ is F_j.

Since each color class is a matching, we will say that W is an *owner* of the set of colors $\mathcal{C}$.

In order to prove our theorem, we show the following lemma.

Lemma 7. *Let $n = q^2+q+1$ with q a natural number, such that Π_q exists. Let K_n be a representation of Π_q and let $\varsigma\colon E(K_n) \to \mathcal{C}$ be a partial edge-coloring of K_n. Suppose that each line L_i of K_n is an owner of the set of colors $\mathcal{C}_i \subseteq \mathcal{C}$. Then ς is complete.*

Proof. Let $\{c_1,c_2\} \subseteq \bigcup_{i=1}^{n} \mathcal{C}_i$. If there is an $i \in \{1,\ldots,n\}$ such that $\{c_1,c_2\} \subseteq \mathcal{C}_i$, since L_i is an owner of $\mathcal{C}_i$, it follows that each $u \in V(L_i)$ is incident with edges colored c_1 and c_2.

If $c_1 \in \mathcal{C}_i$ and $c_2 \in \mathcal{C}_j$ with $i \neq j$, then there is $u \in V(G)$ such that $u = V(L_i) \cap V(L_j)$ and since L_i and L_j are owners of $\mathcal{C}_i$ and $\mathcal{C}_j$, respectively, by the properties of the projective plane, u is incident with edges colored c_1 and c_2. ■

Theorem 8. *Let Π_q be a cyclic projective plane of odd order q with difference set D, $n = q^2+q+1$ and $C_n(J)$ a circulant graph such that $J \subseteq \left\{1,\ldots,\left\lfloor \frac{n}{2} \right\rfloor\right\}$. If the chords of D have lengths of J of L_0 and they are the union of t matchings $(1 \le t \le q)$, then*

$$tn \le \alpha_1(C_n(J)).$$

Proof. In order to prove this theorem, we exhibit a proper and complete edge-coloring of $C_n(J)$ for $n = q^2+q+1$, with $k = tn$ colors. Since J is a subset of $\left\{1,\ldots,\left\lfloor \frac{n}{2} \right\rfloor\right\}$ and $\left\lfloor \frac{n}{2} \right\rfloor = \binom{q+1}{2}$, then $t = 2|J|/(q+1)$. Let $\mathcal{C}$ be a set of k colors, and let $\{\mathcal{C}_0,\ldots,\mathcal{C}_{n-1}\}$ be a partition of $\mathcal{C}$ such that $\mathcal{C}_i$ is a set of t colors for all $i \in \{0,\ldots,n-1\}$.

Let $V(C_n(J))$ be representing the points of a cyclic plane Π_q, and let $\{L_0,\ldots,L_{n-1}\}$ be the set of lines of induced in $V(C_n(J))$ such that $V(L_0) = D$.

To color the edges of $V(C_n(J))$ with $\varsigma\colon V(C_n(J)) \to [k]$, we assign partial colorings of Type M_i using the colors of $\mathcal{C}_i$ to the lines of L_i for all $i \in \{0,\ldots,n-1\}$.

Since each color class is a matching, the coloring is proper. Since each line L_i is an owner of each color of $\mathcal{C}_i$ $(1 \le i \le n)$ by Lemma 7, it follows that the resultant edge-coloring $\varsigma := \bigcup_{i=1}^{n} \varsigma_i$ of $V(C_n(J))$ is a complete edge-coloring using k colors. ■

Next, we analyze the upper bounds. Note that a circulant graph $G = C_n(J)$ is r-regular with $r = 2|J|$, therefore, it has size $n\binom{r}{2}$ and by Equation 3, we have that

$$\alpha_1(G) \le \left\lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + nr(r-1)} \right\rfloor .$$

This equation works better with some graphs (normally with sparse graphs, i.e., those for which its size is much less than $|V|^2$) but for these graphs we give a different approach to obtain an upper bound, first introduced for complete graphs, see [17]. Let $f_{n,r}(x)$ and $g_{n,r}(x)$ be functions that count the maximum number of chromatic classes in two different ways, where x is the size of the smallest chromatic class defined as follows.

$$f_{n,r}(x) = \frac{nr}{2x} \text{ and } g_{n,r}(x) = \begin{cases} 2x(r-1)+1 & \text{if } r < n-2x, \\ x(n-2x+r-1)+1 & \text{if } r \ge n-2x. \end{cases}$$

Since $f_{n,r}$ is a decreasing function and $g_{n,r}$ is increasing function, we get that

$$\alpha_1(G) \le \max\left\{\min\{\lfloor f_{n,r}(x)\rfloor,\ g_{n,r}(x)\colon x \in \mathbb{N}\}\right\},$$

and thus,

$$\alpha_1(G) \le \max\left\{\min\{f_{n,r}(x),\ g_{n,r}(x)\colon x \in \mathbb{R}\}\right\}.$$

Theorem 9. *Let $n = q^2 + q + 1$ and $C_n(J)$ be a circulant r-regular graph such that $r = (q+1)t$ with $1 \le t \le q$. If there exists Π_q a cyclic projective plane of order odd q such that $J \subseteq \left\{1, \dots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$, the chords of D have lengths of J of L_0 and they are the union of t matchings, then*

$$\alpha_1(C_n(J)) = tn.$$

Proof. Since the result is true for complete graphs, assume $1 \le t \le q-1$. By Theorem 8, $tn \le \alpha_1(C_n(J))$ and $\alpha_1(C_n(J)) \le f_{n,r}(x) = \frac{n(q+1)t}{2x}$ and then $x \le \frac{q+1}{2}$.

Suppose $x = \frac{q+1}{2}$. Then $f_{n,r}(x) = tn$ and $g_{n,r}(x) = ((q+1)t-1)(q+1) + 1 = tn + tq - q$ since $(q+1) \le t(q+1) \le q^2 - 1 < q^2 = n - 2x$. Hence $\min\{\lfloor f_{n,r}(x)\rfloor, g_{n,r}(x)\} = f_{n,r}(x) = tn = q^2t + qt + t$.

Now, suppose $x = \frac{q-1}{2}$. Then $f_{n,r}(x) = tn\frac{q+1}{q-1} = \frac{q^3t+2q^2t+2qt+t}{q-1}$ and $g_{n,r}(x) = ((q+1)t-1)(q-1)+1 = \frac{q^3t-q^2t-qt+t-(q^2-3q+2)}{q-1}$ (newly, since $(q+1) \le t(q+1) \le q^2 - 1 < q^2 = n - 2x$). Hence $\min\{\lfloor f_{n,r}(x)\rfloor, g_{n,r}(x)\} = g_{n,r}(x) = q^2t - t - q + 2$ since $q \ge 3$.

Therefore,

$$\alpha_1(C_n(J)) \le \max\left\{q^2t + qt + t, q^2t - t - q + 2\right\} = q^2t + qt + t = tn,$$

i.e.,

$$\alpha_1(C_n(J)) = tn. \qquad \blacksquare$$

For example, $\alpha_1(C_n(1,2)) = \alpha_1(C_n(3,6)) = \alpha_1(C_n(4,5)) = 13$, $\alpha_1(C_n(3,4,5,6)) = \alpha_1(C_n(1,2,4,5)) = \alpha_1(C_n(1,2,3,6)) = 26$ and $\alpha_1(C_n(1,2,3,4,5,6)) = 39$.

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References

[1] G. Araujo-Pardo, J.J. Montellano-Ballesteros, M. Olsen and C. Rubio-Montiel, *The diachromatic number of digraphs*, Electron. J. Combin. **25** (2018) #P3.51
doi:10.37236/7807

[2] G. Araujo-Pardo, J.J. Montellano-Ballesteros, C. Rubio-Montiel and R. Strausz, *On the pseudoachromatic index of the complete graph* III, Graphs Combin. **34** (2018) 277–287.
doi:10.1007/s00373-017-1872-6

[3] G. Araujo-Pardo, J. J. Montellano-Ballesteros and R. Strausz, *On the pseudoachromatic index of the complete graph*, J. Graph Theory **66** (2011) 89–97.
doi:10.1002/jgt.20491

[4] G. Araujo-Pardo, J.J. Montellano-Ballesteros, R. Strausz and C. Rubio-Montiel, *On the pseudoachromatic index of the complete graph* II, Bol. Soc. Mat. Mex. **20** (2014) 17–28.
doi:10.1007/s40590-014-0007-9

[5] G. Araujo-Pardo and C. Rubio-Montiel, *On $\omega\psi$-perfect graphs*, Ars Combin. **141** (2018) 375–387.

[6] F. Bories and J. Jolivet, *On complete colorings of graphs*, in: Recent Advances in Graph Theory, Proc. Second Czechoslovak Sympos., Prague, 1974, (Academia, Prague, 1975) 75–87.

[7] A. Bouchet, *Indice achromatique des graphes multiparti complets et réguliers*, Cahiers Centre Études Rech. Opér. **20** (1978) 331–340.

[8] D.M. Burton, Elementary Number Theory, Second Edition (W.C. Brown Publishers, Dubuque, IA, 1989).

[9] N. Cairnie and K. Edwards, *Some results on the achromatic number*, J. Graph Theory **26** (1997) 129–136.
doi:10.1002/(SICI)1097-0118(199711)26:3⟨129::AID-JGT3⟩3.0.CO;2-T

[10] G. Chartrand and P. Zhang, Chromatic Graph Theory (Chapman and Hall/CRC Press, New York, 2008).
doi:10.1201/9781584888017

[11] M. Dębski, Z. Lonc and P. Rzążewski, *Achromatic and harmonious colorings of circulant graphs*, J. Graph Theory **87** (2018) 18–34.
doi:10.1002/jgt.22137

[12] K.J. Edwards, *Harmonious chromatic number of directed graphs*, Discrete Appl. Math. **161** (2013) 369–376.
doi:10.1016/j.dam.2012.09.003

[13] F. Harary, S. Hedetniemi and G. Prins, *An interpolation theorem for graphical homomorphisms*, Port. Math. **26** (1967) 453–462.

[14] P. Hell and D.J. Miller, *Graph with given achromatic number*, Discrete Math. **16** (1976) 195–207.
doi:10.1016/0012-365X(76)90099-6

[15] M. Horňák, *Achromatic index of* K_{12}, Ars Combin. **45** (1997) 271–275.

[16] M. Horňák, Š. Pčola and M. Woźniak, *On the achromatic index of* K_{q^2+q} *for a prime* q, Graphs Combin. **20** (2004) 191–203.
doi:10.1007/s00373-004-0550-7

[17] R.E. Jamison, *On the edge achromatic numbers of complete graphs*, Discrete Math. **74** (1989) 99–115.
doi:10.1016/0012-365X(89)90202-1

[18] R.E. Jamison, *On the achromatic index of* K_{12}, in: Proceedings of the Twenty-second Southeastern Conference on Combinatorics, Graph Theory and Computing, Baton Rouge, 1991, Congr. Numer. **81** (1991) 143–148.

[19] F. Kárteszi, Introduction to Finite Geometries (North-Holland Publ. Co., New York, 1976).

[20] V. Neumann-Lara, *The dichromatic number of a digraph*, J. Combin. Theory Ser. B **33** (1982) 265–270.
doi:10.1016/0095-8956(82)90046-6

[21] É. Sopena, *Complete oriented colourings and the oriented achromatic number*, Discrete Appl. Math. **173** (2014) 102–112.
doi:10.1016/j.dam.2014.03.015

[22] C. M. Turner, R. Rowley, R. Jamison and R. Laskar, *The edge achromatic number of small complete graphs*, Congr. Numer. **62** (1988) 21–36.

[23] V. Yegnanarayanan, *The pseudoachromatic number of a graph*, Southeast Asian Bull. Math. **24** (2000) 129–136.
doi:10.1007/s10012-000-0129-z

[24] V. Yegnanarayanan, *Graph colourings and partitions*, Theoret. Comput. Sci. **263** (2001) 59–74.
doi:10.1016/S0304-3975(00)00231-0

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