Disproof of a Conjecture of Neumann-Lara

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Abstract

We disprove the following conjecture due to Víctor Neumann-Lara: for every pair (r, s) of integers such that $r \geqslant s \geqslant 2$, there is an infinite set of circulant tournaments T such that the dichromatic number and the cyclic triangle free disconnection of T are equal to r and s, respectively. Let $\mathcal{F}_{r,s}$ denote the set of circulant tournaments T with $dc(T) = r$ and $\overrightarrow{\omega}_3(T) = s$. We show that for every integer $s \geqslant 4$ there exists a lower bound $b(s)$ for the dichromatic number r such that $\mathcal{F}_{r,s} = \emptyset$ for every $r < b(s)$. We construct an infinite set of circulant tournaments T such that $dc(T) = b(s)$ and $\overrightarrow{\omega}_3(T) = s$ and give an upper bound $B(s)$ for the dichromatic number r such that for every $r \geqslant B(s)$ there exists an infinite set $\mathcal{F}_{r,s}$ of circulant tournaments. Some infinite sets $\mathcal{F}_{r,s}$ of circulant tournaments are given for $b(s) < r < B(s)$.

Keywords: Circulant tournaments; dichromatic number; acyclic disconnection

1 Introduction

The dichromatic number and the acyclic (respectively, cyclic triangle free) disconnection were introduced as measures of the complexity of the cyclic structure of digraphs. A large value of the dichromatic number and, oppositely, a small value of the acyclic disconnection

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express a more complex cyclic structure of a given digraph. Among other papers, see [1, 5, 7, 8, 11, 12, 13, 14, 15] for old and recent results on the study of these parameters as well as open problems. Many variations of colorings of digraphs have been extensively studied as the oriented chromatic number and the oriented game chromatic number. See for example [10] and [18].

We define the *dichromatic number* of a digraph D, denoted by $dc(D)$, as the minimum number of colors in a coloring of the vertices of D such that each chromatic class induces an acyclic subdigraph of D (that is, a subdigraph containing no directed cycles). In this terminology, the notion was introduced by V. Neumann-Lara in [12]. Even before, the dichromatic number for graphs and digraphs is defined by P. Erdős in [4] (pp. 16–20) reporting some results obtained in a joint work with V. Neumann-Lara. In this earlier paper, article [12] is mentioned to be in preparation.

On the other hand, the *acyclic* (respectively, *cyclic triangle free* or briefly, the $\overrightarrow{C}_3$*-free*) *disconnection* of a digraph D, denoted by $\overrightarrow{\omega}(D)$ (respectively, $\overrightarrow{\omega}_3(D)$), is defined to be the maximum number of colors in a coloring of the vertices of D such that no directed cycle is properly colored (respectively, no directed 3-cycle is 3-colored). We recall that in a proper coloring of the vertices of a digraph D, consecutive vertices of a directed cycle receive different colors. A digraph D (in particular, a tournament T) is said to be *tight* if $\overrightarrow{\omega}_3(D) = 2$ (respectively, $\overrightarrow{\omega}_3(T) = 2$). These definitions first appeared in [13].

In 1999, V. Neumann-Lara posed the following

Conjecture 1 ([13], Conjecture 5.8)**.** For every pair (r, s) of integers such that $r \geqslant s \geqslant 2$, there is an infinite set of regular (circulant) tournaments T such that $dc(T) = r$ and $\overrightarrow{\omega}_3(T) = s$ (respectively, $\overrightarrow{\omega}(T) = s$).

Let T be a regular tournament. We define

$$\begin{aligned} \mathcal{F}_{r,s} &= \{T : dc(T) = r, \overrightarrow{\omega}_3\,(T) = s\} \quad \text{and} \\ \widetilde{\mathcal{F}}_{r,s} &= \{T : dc(T) = r, \overrightarrow{\omega}\,(T) = s\}. \end{aligned}$$

The following theorem will be useful to simplify the proofs.

Theorem 2 ([6], Theorem 19 and Corollary 22)**.** *Every prime circulant tournament T is tight. Moreover, $\overrightarrow{\omega}_3(T) = \overrightarrow{\omega}(T)$ for every circulant tournament T.*

In this paper, we only deal with circulant tournaments. We notice that in virtue of Theorem 2, it suffices to consider $\overrightarrow{\omega}_3(T)$ for every circulant tournament T. If some statement is valid for $\overrightarrow{\omega}_3(T)$ or for a family of type $\mathcal{F}_{r,s}$, it holds for $\overrightarrow{\omega}(T)$ or for a family of type $\widetilde{\mathcal{F}}_{r,s}$.

In [8], the authors positively answer the conjecture for the special case when $r = 3$ and $s = 2$ giving an infinite family of 3-dichromatic tight regular *not* circulant tournaments. However, in this paper the conjecture is disproved in general. We show that for every integer $s \geqslant 4$ there exists a lower bound $b(s)$ for the dichromatic number r such that $\mathcal{F}_{r,s} = \emptyset$ for every $r < b(s)$. We construct an infinite set of circulant tournaments T such that $dc(T) = b(s)$ and $\overrightarrow{\omega}_3(T) = s$. All this is summarized in the main theorem of this paper (see the proof in Section 4).

Theorem 3. *Let T be a circulant tournament such that $dc(T) = r$ and $\overrightarrow{\omega}_3(T) = s$ $(2 \leqslant s \leqslant r)$. Let $b(s) = D_{s-1}$, where $D_s = \left\lceil \frac{3}{2} D_{s-1} \right\rceil$ $(s \geqslant 2$ and $D_1 = 2)$. Then*

$$r \geqslant b(s) = \left\lceil \frac{3}{2} D_{s-2} \right\rceil = \left\lfloor K \left(\frac{3}{2} \right)^{s-1} \right\rfloor,$$

where $K \approx 1.62227$ is an irrational number. Moreover, $\left\{ \overrightarrow{C}_3^{s-2}[\alpha], \alpha \in \mathcal{F}_{2,2} \right\}$ is an infinite set of r-dichromatic circulant tournaments such that $\overrightarrow{\omega}_3 \left(\overrightarrow{C}_3^{s-2}[\alpha] \right) = s$ and $r = b(s)$.

We give an upper bound $B(s)$ for the dichromatic number r such that for every $r \geqslant B(s)$ there exists an infinite set $\mathcal{F}_{r,s}$ of circulant tournaments (see Proposition 23). Some infinite sets $\mathcal{F}_{r,s}$ of circulant tournaments are given for $b(s) < r < B(s)$. The construction of the remaining cases in this interval is an open problem since the tools used in the paper do not apply for them.

Our proofs are strongly based on the techniques developed in [14]. For the usual terminology on digraphs and tournaments used in the paper, see [2, 3, 17].

An extended abstract of some parts of this work, with no proofs, appeared in [9].

2 Preliminaries

Let $D = (V, A)$ be a digraph. For any $v \in V(D)$ we denote by $N^+(v)$ or $N^+(v, D)$ and $N^-(v)$ or $N^-(v, D)$ the out- and in-neighborhood of v in D, respectively. A digraph D is said to be *acyclic* if D contains no directed cycles. A subset $S \subseteq V(D)$ is *acyclic* if the induced subdigraph $D\langle S \rangle$ of D by the set S is acyclic. The maximum cardinality of an acyclic set of vertices of D is denoted by $\beta(D)$.

An r-coloring $\varphi : V(D) \to \{1, 2, \ldots, r\}$ of a digraph D is a surjective function. A subdigraph D' of D is *heterochromatic* or *rainbow* if every pair of vertices of D' receive different colors under φ. A subdigraph D' of D is *properly colored* if every pair of adjacent vertices of D' receive different colors under φ. A subset S of vertices of D that receive the same color under φ is called a *chromatic class* and it is a *singleton* if $|S| = 1$. We say that a r-coloring φ of a digraph D is $\overrightarrow{C}_3$*-free* (respectively, $\overrightarrow{C}$*-free*) if D contains no rainbow cyclic triangles (respectively, no properly colored directed cycles).

Let D and F be digraphs and $\{F_v\}_{v \in V(D)}$ a family of mutually disjoint isomorphic copies of F. The *composition* (or *lexicographic product* $D \circ F$) $D[F]$ of the digraphs D and F is defined by $V(D[F]) = \bigcup_{v \in V(D)} V(F_v)$ and

$$A(D[F]) = \Big[\bigcup_{v \in V(D)} A(F_v) \Big] \cup \big\{ (i, j) : i \in V(F_v),\ j \in V(F_w) \text{ and } (v, w) \in A(D) \big\}.$$

It is easy to prove (and left to the reader) that the composition of digraphs is an associative but not a commutative operation.

Let $\mathbb{Z}_{2m+1}$ be the cyclic group of integers modulo $2m + 1$ $(m \geqslant 1)$ and J a nonempty subset of $\mathbb{Z}_{2m+1} \setminus \{0\}$ such that $|\{-j, j\} \cap J| = 1$ for every $j \in \mathbb{Z}_{2m+1} \setminus \{0\}$ (and therefore

$|J| = m$). A *circulant* (or *rotational*) *tournament* $\overrightarrow{C}_{2m+1}(J)$ is defined by $V(\overrightarrow{C}_{2m+1}(J)) = \mathbb{Z}_{2m+1}$ and

$$A(\overrightarrow{C}_{2m+1}(J)) = \{(i,j) : i,j \in \mathbb{Z}_{2m+1} \text{ and } j - i \in J\}.$$

Recall that the circulant tournaments are regular and their automorphism groups are vertex–transitive. Let $[m] = \{1, 2, \ldots, m\}$. We denote by $\overrightarrow{C}_{2m+1}\langle\emptyset\rangle$ and $\overrightarrow{C}_{2m+1}\langle j\rangle$ the circulant tournaments $\overrightarrow{C}_{2m+1}(J)$ where $J = [m]$ and $J = ([m] \setminus \{j\}) \cup \{-j\}$, respectively.

Observe that $\overrightarrow{C}_3 = \overrightarrow{C}_3\langle\emptyset\rangle$. Moreover, $\overrightarrow{C}_3 \in \mathcal{F}_{2,2}$.

Proposition 4 ([13], Proposition 3.3)**.** *The composition of two circulant tournaments is a circulant tournament.*

The above proposition holds for circulant digraphs in general. A (circulant) tournament T is *prime* (or *simple*) if it is not isomorphic to a composition of (circulant) tournaments.

Proposition 5 ([15], Theorem 1)**.** *Let $m \in \mathbb{N}$. Then $dc(\overrightarrow{C}_{2m+1}\langle\emptyset\rangle) = 2$.*

Proposition 6 ([13], Proposition 4.4(i), Theorem 4.11)**.** *Let $m \in \mathbb{N}$.*

(i) $\overrightarrow{\omega}_3(\overrightarrow{C}_{2m+1}\langle\emptyset\rangle) = \overrightarrow{\omega}(\overrightarrow{C}_{2m+1}\langle\emptyset\rangle) = 2$.

(ii) $\overrightarrow{\omega}_3(\overrightarrow{C}_{2m+1}\langle m-1\rangle) = \overrightarrow{\omega}(\overrightarrow{C}_{2m+1}\langle m-1\rangle) = 2$.

We use the following definition taken from [13]. A digraph D will be said to be $\overrightarrow{\omega}$-*keen* (respectively, $\overrightarrow{\omega}_3$-*keen*) if there is an optimal coloring φ of $V(D)$ (that is, it uses the maximum number of colors), with no properly colored directed cycles (respectively, with no 3-colored $\overrightarrow{C}_3$) of D, having exactly one singleton chromatic class. Notice that no optimal coloring φ of $V(D)$ leaves more than one such a class.

Lemma 7 ([6], Theorems 17,18)**.** *Every circulant tournament is $\overrightarrow{\omega}_3$-keen (respectively, $\overrightarrow{\omega}$-keen).*

This lemma was an important tool to prove Theorem 2 in [6]. It will be later used in other proofs of the paper.

3 Infinite sets of r-dichromatic circulant tournaments for every $r \geqslant 2$ and $s = 2$

This section is devoted to the construction of infinite families of circulant tournaments T such that $dc(T) = r$ and $\overrightarrow{\omega}_3(T) = 2$ (that is, tight circulant tournaments), where $r \geqslant 2$.

Let $\overrightarrow{C}_{n(p+1)+1}(J)$ be the set of circulant tournaments defined in [1], where

$$\begin{aligned} J &= \{1,2,\ldots,p\} \cup \{p+2, p+3, \ldots, 2p-t\} \cup \\ &\qquad \{2p+3, \ldots, 3p-2t\} \cup \ldots \cup \{(n-1)p+n\} \\ &= \bigcup_{i=0}^{n-1} \{ip + (i+1), \ldots, ip + p - it\} \end{aligned}$$

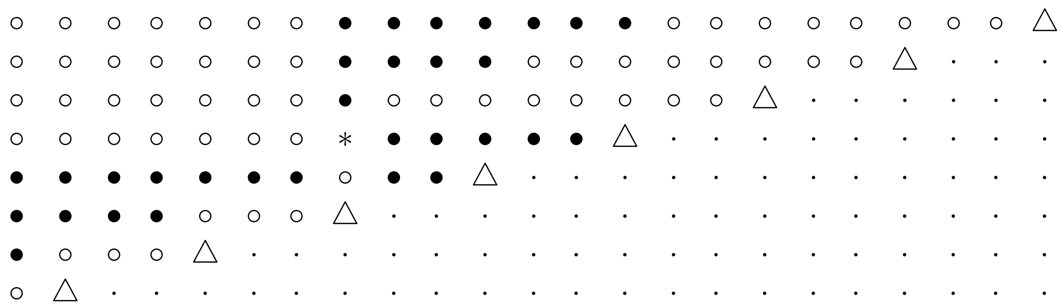

Figure 1: The tournament $\mathbf{H}_{22,2}$ and the out-neighborhood of $m_{ij} = m_{4,8}$.

and $p = (n-1)(t+1) + 1$, $n \geqslant 2$, $t \geqslant 0$.

To understand the structure of these tournaments it is convenient to establish a one to one correspondence between the out-neighborhood of a vertex and the following subset $H_{p,t}$ of entries m_{ij} in a $M_{n\times(p+1)}$ matrix:

$$H_{p,t} = \left\{m_{ij} \in M_{n\times(p+1)} : (i-1)(t+1) + j \leqslant p\right\}.$$

Let $v \in V(\overrightarrow{C}_{n(p+1)+1}(J))$, and $i, j \in \mathbb{N}$ such that $v = (p+1)(i-1) + j$, with $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant p$. The function $\psi : N^+(v) \to H_{p,t}$ defined by $\psi(w) = m_{ij}$ is clearly a bijection. Since $\overrightarrow{C}_{n(p+1)+1}(J)$ is vertex-transitive, we have that $J = N^+(0)$. Let $\mathbf{H}_{p,t}$ be the induced subtournament by $N^+(0)$ in $\overrightarrow{C}_{n(p+1)+1}(J)$. Notice that the induced subtournament by $N^+(v)$ is isomorphic to $\mathbf{H}_{p,t}$ for every $v \in V(\overrightarrow{C}_{n(p+1)+1}(J))$.

We define the out-neighborhood of each vertex $m_{ij} \in H_{p,t}$ as the union of three disjoint sets of vertices in $H_{p,t}$, specifically $N^+(m_{ij}, \mathbf{H}_{p,t}) = A_{ij} \cup B_{ij} \cup C_{ij}$, where

$$\begin{aligned}
A_{ij} = \{m_{kl} \in M_{n\times(p+1)} :\ & (k-1)(s+1) + l - j \leqslant (i-2)(s+1), \\
& k \geqslant 1, l \geqslant j\}, \\
B_{ij} = \{m_{kl} \in M_{n\times(p+1)} :\ & (k-i)(s+1) + l - j - 1 < (n-i)(s+1) - j + r, \\
& k \geqslant i, l \geqslant j+1\}, \\
C_{ij} = \{m_{kl} \in M_{n\times(p+1)} :\ & (k-i-1)(s+1) + l - 1 \leqslant j - 2, \\
& k \geqslant i+1, l \geqslant 1\}.
\end{aligned}$$

By Lemma 1 of [1], the subtournaments induced by A_{ij}, B_{ij} and C_{ij} are the vertex-disjoint tournaments $\mathbf{H}_{1+(i-2)(t+1),t}$, $\mathbf{H}_{p-(i-1)(t+1)-j,t}$ and $\mathbf{H}_{j-1,t}$, respectively. In Figure 1, the entry $m_{4,8}$ is the point $*$, the vertices in the out-neighborhood of $*$ in $\mathbf{H}_{22,2}$ are the $\bullet$ points partitioned into the sets $A_{4,8}$, $B_{4,8}$ and $C_{4,8}$ (the triangles that appear at the top, to the right and to the left of $*$, respectively). Notice that the vertices in the in-neighborhood of $*$ in $\mathbf{H}_{22,2}$ are the $\circ$ points and the $\cdot$ and $\triangle$ points are not vertices of $\mathbf{H}_{22,2}$.

Lemma 8 ([1], Theorem 5). *$dc\left(\overrightarrow{C}_{n(p+1)+1}(J)\right) = n + 1$.*

Theorem 9. *$\overrightarrow{C}_{n(p+1)+1}(J)$ is tight (that is, $\overrightarrow{\omega}_3(\overrightarrow{C}_{n(p+1)+1}(J)) = 2$).*

Proof. Let $T \cong \overrightarrow{C}_{n(p+1)+1}(J)$. For a contradiction, suppose that T is not tight, that is, there exists a $\overrightarrow{C}_3$-free 3-coloring $\varphi : V(T) \to \{blue, green, red\}$ of T. By Lemma 7, T is $\overrightarrow{\omega}_3$-keen and therefore there is exactly one singleton chromatic class. Since T is vertex-transitive, we may assume that $\varphi(0) = blue$, and 0 is the only vertex of color *blue*. Suppose that $\varphi(1) = green$ ($\varphi(m_{11}) = green$ in matrix notation). Since $(0, 1, i)$ is a cyclic triangle for every $i \in \{kp - (k-1)t + 1 : 1 \leqslant k \leqslant n\}$ (these cyclic triangles correspond to $(0, m_{11}, \triangle)$ in the matrix $M_{n\times(p+1)}$) and φ is $\overrightarrow{C}_3$-free, then $\varphi(i) = green$. Let us define

$$N_k = \{(k-1)p + k, (k-1)p + k + 1, \ldots, kp - (k-1)t\}$$

for $1 \leqslant k \leqslant n$ (N_k can be viewed as the set of elements $\circ$ and $\bullet$ of the k-th row of the matrix $M_{n\times(p+1)}$). Observe that $(0, j, kp - (k-1)t + 1)$ (respectively, $(0, \circ, \triangle)$ or $(0, \bullet, \triangle)$) is a cyclic triangle for every $j \in N_k$. Using again that φ is $\overrightarrow{C}_3$-free, we have that $\varphi(j) = green$ for every $j \in N_k$. Therefore $N^+(0)$ is monochromatic. Observe that $kp - (k-1)t + 1 \in N^-(0)$ (elements of type $kp - (k-1)t + 1$ correspond to $\triangle$ in the matrix) and $\varphi(kp - (k-1)t + 1) = green$. Analogously, we can conclude that $N^-(0)$ is monochromatic. We have only used two colors, so φ is not surjective, a contradiction to the initial assumption. $\square$

Observe that $\overrightarrow{\omega}(\overrightarrow{C}_{n(p+1)+1}(J)) = 2$ by Theorem 2.

Corollary 10. *For every $r \geqslant 2$ there exists an infinite set $\mathcal{F}_{r,2}$ of tight r-dichromatic circulant tournaments.*

Proof. By Propositions 5 and 6(i), the infinite set $\mathcal{F}_{2,2} = \left\{\overrightarrow{C}_{2n+1}\langle\emptyset\rangle : n \in \mathbb{N}\right\}$ is the desired set for $r = 2$. By Lemma 8 and Theorem 9, for every $r \geqslant 3$ the infinite set

$$\mathcal{F}_{r+1,2} = \left\{\overrightarrow{C}_{r(p+1)+1}(J) : p = (r-1)(t+1) + 1, t \geqslant 0\right\}$$

provides the remaining cases. $\square$

4 Infinite sets for $s \geqslant 3$

Let $H = (V, E)$ a finite hypergraph. A hypergraph H is *t-uniform* (or simply, a *t-graph*) if every edge of H has cardinality t. A hypergraph H is called *circulant* if it has an automorphism which is a cyclic permutation of $V(H)$. If $t \leqslant m$, the circulant t-graph $\Lambda_{m,t}$ is defined by $V(\Lambda_{m,t}) = \mathbb{Z}_m$ and $E(\Lambda_{m,t}) = \{\alpha_j : j \in \mathbb{Z}_m\}$ where

$$\alpha_j = \{j, j+1, \ldots, j+t-1\}$$

for $j \in \mathbb{Z}_m$.

We denote by $\beta(T)$ the maximum cardinality of an acyclic set of vertices of a tournament T. In [12] (see Theorem 8), it was proved that $dc(T[U]) \geqslant dc(T) + dc(U) - 1$. Using this result as well as Propositions 32(iii) and 34 and Corollary 43 of [14], it is not hard to establish the following proposition that will be useful in the proof of Theorem 3.

Proposition 11. *Let T and U be circulant tournaments such that T has order $2m+1$ and U is an r-dichromatic tournament. Then*

$$dc\left(T\left[U\right]\right) \geqslant \left\lceil \frac{r\left(2m+1\right)}{m+1} \right\rceil .$$

Moreover, if $\beta\left(T\right)=t$ and T contains an isomorphic copy of a t-graph $\Lambda_{2m+1,t}$ (where $V\left(\Lambda_{2m+1,t}\right)=\mathbb{Z}_{2m+1}$, $E\left(\Lambda_{2m+1,t}\right)=\left\{\alpha_j : j\in\mathbb{Z}_{2m+1}\right\}$, $\alpha_j=\left\{j,j+1,\ldots,j+t-1\right\}$ and sums are taken modulo $2m+1$), then

$$dc\left(T\left[U\right]\right) = \left\lceil \frac{r\left(2m+1\right)}{t} \right\rceil .$$

Proposition 12 ([13], Proposition 3.6(i))**.** *Let T be a $\overrightarrow{\omega}_3$-keen (respectively, $\overrightarrow{\omega}$-keen) tournament and U an arbitrary tournament. Then*

$$\begin{aligned}\overrightarrow{\omega}_3\left(T\left[U\right]\right) &= \overrightarrow{\omega}_3\left(T\right)+\overrightarrow{\omega}_3\left(U\right)-1\\ \left(\overrightarrow{\omega}\left(T\left[U\right]\right)\right. &= \left.\overrightarrow{\omega}\left(T\right)+\overrightarrow{\omega}\left(U\right)-1\right).\end{aligned}$$

Proposition 13 ([16], Corollary 1)**.** *Consider the recurrence relation $D_n=\left\lceil\frac{3}{2}D_{n-1}\right\rceil$ ($n\geqslant 1$ and $D_0=1$), then*

$$D_n=\left\lfloor K\left(\frac{3}{2}\right)^n\right\rfloor \qquad \left(n=1,2,\ldots\right),$$

where $K\approx 1.62227$ is an irrational number.

The previous recurrence relation appears in the solution of the legendary Josephus Flavius problem. It is the classical case when n Jews formed in a circle decide to kill every third remaining person until no one is left (the last survivor must commit suicide). In the story, there were 40 Jewish soldiers trapped in a cave by the Roman army who chose suicide rather than be captured. For more details about the mathematical problem see [16].

The following proposition is a consequence of Theorem 2.

Proposition 14. *Let T be a circulant tournament with $\overrightarrow{\omega}_3\left(T\right)=s\geqslant 3$. There exist $s-1$ tight circulant tournaments $T_1,T_2,\ldots,T_{s-1}$ such that $T\cong T_1\left[T_2\left[\ldots\left[T_{s-1}\right]\right]\right]$.*

Proof. We proceed by induction on s. If $s=3$, then by Theorem 2, T is a composition of two circulant tournaments T_1 and T_2, that is, $T\cong T_1[T_2]$. Observe that if $\overrightarrow{\omega}_3(T_1)=s_1\geqslant 2$ and $\overrightarrow{\omega}_3(T_2)=s_2\geqslant 2$, then

$$\overrightarrow{\omega}_3(T_1[T_2])=s_1+s_2-1=3,$$

$s_1=s_2=2$ and both T_1 and T_2 are tight circulant tournaments.

Suppose that the claim is valid for every $3\leqslant\overrightarrow{\omega}_3(T)=s'<s$. By the induction hypothesis, there exist $s'-1$ tight circulant tournaments $T_1,T_2,\ldots,T_{s'-1}$ such that

$$T\cong T_1[T_2[\ldots[T_{s'-1}]]].$$

Let $\overrightarrow{\omega}_3(T) = s$. Hence, T is a composition of two circulant tournaments U and W, that is $T \cong U[W]$. Let $\overrightarrow{\omega}_3(U) = t_1 \geqslant 2$ and $\overrightarrow{\omega}_3(W) = t_2 \geqslant 2$. Since

$$3 \leqslant \overrightarrow{\omega}_3(U[W]) = t_1 + t_2 - 1 = s,$$

then $2 \leqslant t_1 < s$ and $2 \leqslant t_2 < s$. By the induction hypothesis, there exist $t_1 - 1$ tight circulant tournaments $U_1, U_2, \ldots, U_{t_1-1}$ such that $U \cong U_1[U_2[\ldots[U_{t_1-1}]]]$, and $t_2 - 1$ tight circulant tournaments $W_1, W_2, \ldots, W_{t_2-1}$ such that $W \cong W_1[W_2[\ldots[W_{t_2-1}]]]$, then

$$T \cong U[W] \cong (U_1[U_2[\ldots[U_{t_1-1}]]])[W_1[W_2[\ldots[W_{t_2-1}]]]].$$

Using the associative property of the composition, we have that

$$T \cong U_1[U_2[\ldots[U_{t_1-1}[W_1[W_2[\ldots[W_{t_2-1}]]]]]]].$$

Therefore, T is the composition of $t_1 - 1 + t_2 - 1 = s - 1$ tight circulant tournaments. $\square$

Let $s \in \mathbb{N}$ and define $\overrightarrow{C}_3^1 = \overrightarrow{C}_3$ and $\overrightarrow{C}_3^s = \overrightarrow{C}_3\left[\overrightarrow{C}_3^{s-1}\right]$ for every $s \geqslant 2$. Observe that in virtue of Proposition 12 and Lemma 7, $\overrightarrow{\omega}_3\left(\overrightarrow{C}_3^s\right) = s + 1$.

We prove the main theorem of this paper.

Proof. (**Theorem 3**)

Let U be an r-dichromatic tournament. Since the composition of tournaments is associative and by the first inequality of Proposition 11, we have that

$$dc\,(T\,[U]) \geqslant \left\lceil \frac{r\,(2m+1)}{m+1} \right\rceil.$$

Therefore

$$dc\,(T\,[U]) \geqslant \left\lceil 2r - \frac{r}{m+1} \right\rceil \geqslant \left\lceil \frac{3}{2} r \right\rceil.$$

It follows that $dc\,(T\,[U]) \geqslant \left\lceil \frac{3}{2} r \right\rceil$ for every r-dichromatic tournament U.

Now let T be a circulant tournament such that $\overrightarrow{\omega}_3\,(T) = s$. By Proposition 14, there exist $T_1, T_2, \ldots, T_{s-1}$ such that $T \cong T_1\,[T_2\,[\ldots[T_{s-1}]]]$. Let $dc\,(T_{s-1}) = r \geqslant 2 = \left\lceil \frac{3}{2} \right\rceil$, then

$$dc\,(T) \geqslant \underbrace{\left\lceil \frac{3}{2} \left\lceil \frac{3}{2} \left\lceil \ldots \left\lceil \frac{3}{2} r \right\rceil \right\rceil \right\rceil \right\rceil}_{s-2} \geqslant \underbrace{\left\lceil \frac{3}{2} \left\lceil \frac{3}{2} \left\lceil \ldots \left\lceil \frac{3}{2} \right\rceil \right\rceil \right\rceil \right\rceil}_{s-1} = D_{s-1}.$$

By Proposition 13, $dc\,(T) \geqslant \left\lfloor K \left(\frac{3}{2}\right)^{s-1} \right\rfloor$.

Let $T \in \left\{ \overrightarrow{C}_3^{s-2}\,[\alpha], \alpha \in \mathcal{F}_{2,2} \right\}$. Since $\overrightarrow{C}_3$ and α are tight, then $\overrightarrow{\omega}_3\,(T) = s$. Observe that $2 = \left\lceil \frac{3}{2} \right\rceil$, then

$$dc\,(T) = \underbrace{\left\lceil \frac{3}{2} \left\lceil \frac{3}{2} \left\lceil \ldots \left\lceil \frac{3}{2} \right\rceil \right\rceil \right\rceil \right\rceil}_{s-1} = D_{s-1}.$$

By Proposition 13, $dc\,(T) = \left\lfloor K \left(\frac{3}{2}\right)^{s-1} \right\rfloor = b(s)$. $\square$

Remark 15. Notice that $b(s)$ is a lower bound for $dc(T) = r$ when $\overrightarrow{\omega}_3(T) = s$ and there are no r-dichromatic circulant tournaments for which $r < b(s)$.

Let D be a digraph. The hypergraph $H_1(D)$ is defined by $V(H_1(D)) = V(D)$ and

$$E(H_1(D)) = \{U \subseteq V(D) : U \text{ is a maximal acyclic set}\}.$$

Lemma 16 ([14], Proposition 41(i))**.** *Let $m \geqslant 2$. Then*

(i) $H_1(\overrightarrow{C}_{2m+1}\langle\emptyset\rangle) \supseteq \Lambda_{2m+1,m+1}$ *and*

(ii) $\beta(\overrightarrow{C}_{2m+1}\langle\emptyset\rangle) = m + 1$.

We prove a similar result to the previous lemma for the family of circulant tournaments $\overrightarrow{C}_{2m+1}\langle m-1\rangle$ with $m \geqslant 2$.

Lemma 17. *Let $m \geqslant 2$. Then*

(i) $H_1(\overrightarrow{C}_{2m+1}\langle m-1\rangle) \supseteq \Lambda_{2m+1,m-1}$ *and*

(ii) $\beta(\overrightarrow{C}_{2m+1}\langle m-1\rangle) = m - 1$.

Proof. (i) Since $\overrightarrow{C}_{2m+1}\langle m-1\rangle$ is vertex-transitive and $\{0, 1, \ldots, m-2\}$ is an acyclic vertex subset of cardinality $m - 1$, the inclusion is valid.

(ii) Clearly, $\beta(\overrightarrow{C}_{2m+1}\langle m-1\rangle) \geqslant m - 1$. Let U be a vertex set of maximum cardinality, such that $\langle U\rangle$ is isomorphic to the acyclic subtournament of order $|U|$. Since $\overrightarrow{C}_{2m+1}\langle m-1\rangle$ is a vertex-transitive tournament, we may assume that 0 is the source of U. In the out-neighborhood of 0 we have the following sequence:

$$\begin{array}{rcl}
N^+(1) & = & \{2, 3, \ldots, m-2\}, \\
N^+(2) & = & \{3, 4, \ldots, m-2, m, m+2\}, \\
N^+(3) & = & \{4, 5, \ldots, m-2, m\}, \\
N^+(4) & = & \{5, 6, \ldots, m-2, m, m+2\}, \\
 & \vdots & \\
N^+(m-2) & = & \{m, m+2\}, \\
N^+(m) & = & \{1, m+2\}, \\
N^+(m+2) & = & \{1, 3\}.
\end{array}$$

Therefore, if $|U| \geqslant m$, the second source is 2, and $U = \{0, 2, 3, \ldots, m-2, m, m+2\}$. But $(m+2, 3, 4, m+2)$ is a cyclic triangle in $\overrightarrow{C}_{2m+1}\langle m-1\rangle$. Hence, it follows that $\beta(\overrightarrow{C}_{2m+1}\langle m-1\rangle) = m - 1$. ☐

As a consequence of the second equality of Proposition 11 and Lemma 17, we have the following

Corollary 18. *Let α be a r-dichromatic tournament, then*

$$dc\left(\overrightarrow{C}_{2m+1}\langle m-1\rangle[\alpha]\right)=\left\lceil\frac{r\ (2m+1)}{m-1}\right\rceil.$$

Moreover, if $m\geqslant 3r+1$, then $dc\left(\overrightarrow{C}_{2m+1}\langle m-1\rangle[\alpha]\right)=2r+1$.

In what follows, we will recursively construct r-dichromatic circulant tournaments with fixed $\overrightarrow{C}_3$-free disconnection. For this purpose, we define the following functions that will describe the growth of the dichromatic number by the composition of circulant tournaments.

Let f_i', $f_i:\mathbb{N}^2\to\mathbb{N}$ ($i\in\{0,1,2\}$) be functions defined by

$$\begin{array}{lll} f_0'(q,m)=q(m+1)-1, & f_0(q,m)=q(2m+1)-1 & (q\geqslant 1, m\geqslant 2),\\ f_1'(q,m)=qm+1, & f_1(q,m)=q(2m+1)+3 & (q\geqslant 1, m\geqslant 3),\\ f_2'(q,m)=2q+m, & f_2(q,m)=3q+m+1 & (q\geqslant 1, m\in\{1,2\}).\end{array}$$

We define the (infinite) digraphs D_i for $i\in\{0,1,2\}$ as follows:

$$\begin{array}{rcl} V(D_i) & = & \{v\in\mathbb{N}:v\geqslant 3\},\\ A(D_i) & = & \{(f_i'(q,m),f_i(q,m))\}.\end{array}$$

Let $D=D_0\cup D_1\cup D_2$. Clearly, D_0, D_1 and D_2 are arc-disjoint and acyclic. We emphasize that $(u,v)\in A(D)$ if and only if $u=f_i'(q,m)$ and $v=f_i(q,m)$ ($i\in\{0,1,2\}$) for some positive integers q and m such that $q\geqslant 1$ and

(i) $m\geqslant 2$ if $i=0$,

(ii) $m\geqslant 3$ if $i=1$ and

(iii) $1\leqslant m\leqslant 2$ if $i=2$.

Furthermore, if T is an r'-dichromatic tournament with $r'=f_i'(q,m)$, then $W_i[T]$ is an r-dichromatic tournament with $r=f_i(q,m)$ ($i\in\{0,1,2\}$), where $W_0\cong\overrightarrow{C}_{2m+1}\langle\emptyset\rangle$, $W_1\cong\overrightarrow{C}_{2m+1}\langle m\rangle$ and $W_2\cong\overrightarrow{C}_3$. Finally, if $r\equiv 1(\text{mod } 3)$, then by the definition of $f_2(q,m)$ it follows that $dc\left(\overrightarrow{C}_3[T]\right)\neq r$ for every tournament T.

We point out that f_i, f_i' and D_i for $i\in\{0,1\}$ were defined by V. Neumann-Lara in [14], where more details can be found.

Lemma 19 ([14], Lemma 64)**.** *For each positive integer $n\geqslant 3$, $n\neq 7$, there is a directed path in $D=D_0\cup D_1$ from a vertex in $S=\{3,4,5,11,15,23\}$ to n.*

Lemma 20. *For each positive integer $n\geqslant 3$, there is a directed path in $D=D_0\cup D_1\cup D_2$ from a vertex in $S=\{3,4,5,7\}$ to n.*

Proof. A consequence of Lemma 19 and observing that $(7,11),(10,15),(15,23) \subset A(D_2)$ and $(7,13) \in A(D_0)$. Note that by Lemma 8, we have that $\overrightarrow{C}_{30t+43}(J)$ is a 7-dichromatic tournament for every $t \geqslant 0$. □

Proposition 21. *Let $r \geqslant 2$ and $s \geqslant 2$. Then*

(i) $\mathcal{F}_{2r,s+1} = \left\{\overrightarrow{C}_{2r+1}\langle\emptyset\rangle[\alpha] : \alpha \in \mathcal{F}_{r,s}\right\}$.

(ii) $\mathcal{F}_{2r+1,s+1} = \left\{\overrightarrow{C}_{2(3r+1)+1}\langle 3r\rangle[\alpha] : \alpha \in \mathcal{F}_{r,s}\right\}$.

(iii) $\mathcal{F}_{3r,s+1} = \left\{\overrightarrow{C}_{3}[\alpha] : \alpha \in \mathcal{F}_{2r,s}\right\}$.

(iv) $\mathcal{F}_{3r+2,s+1} = \left\{\overrightarrow{C}_{3}[\alpha] : \alpha \in \mathcal{F}_{2r+1,s}\right\}$.

Proof. We proceed case by case.

(i) Let $\alpha \in \mathcal{F}_{r,s}$. By Proposition 11 and Lemma 16, if $r \leqslant m$, then

$$dc\left(\overrightarrow{C}_{2m+1}\langle\emptyset\rangle[\alpha]\right) = \left\lceil \frac{r(2m+1)}{m+1} \right\rceil = \left\lceil 2r - \frac{r}{m+1} \right\rceil = 2r.$$

Hence, $m = r$ and by Propositions 12 and 6(i), $\overrightarrow{\omega}_3\left(\overrightarrow{C}_{2r+1}\langle\emptyset\rangle[\alpha]\right) = s+1$.

(ii) Let $\alpha \in \mathcal{F}_{r,s}$. By Corollary 18, if $m \geqslant 3r+1$, then

$$dc\left(\overrightarrow{C}_{2m+1}\langle m-1\rangle[\alpha]\right) = 2r+1.$$

In this case the equality holds. By Proposition 6(ii) and Lemma 17,

$$\overrightarrow{\omega}_3\left(\overrightarrow{C}_{2m+1}\langle m-1\rangle[\alpha]\right) = s+1.$$

(iii) Let $\alpha \in \mathcal{F}_{2r,s}$. By Proposition 11, $dc\left(\overrightarrow{C}_3[\alpha]\right) = \left\lceil \frac{2r\cdot 3}{2} \right\rceil = 3r$. On the other hand, since $\overrightarrow{C}_3 \in \mathcal{F}_{2,2}$, $\overrightarrow{\omega}_3\left(\overrightarrow{C}_3[\alpha]\right) = s+1$.

(iv) Let $\alpha \in \mathcal{F}_{2r+1,s}$. By Proposition 11, $dc\left(\overrightarrow{C}_3[\alpha]\right) = \left\lceil \frac{(2r+1)\cdot 3}{2} \right\rceil = 3r+2$. On the other hand, since $\overrightarrow{C}_3 \in \mathcal{F}_{2,2}$, $\overrightarrow{\omega}_3\left(\overrightarrow{C}_3[\alpha]\right) = s+1$. □

Proposition 22. *For every integer $r \geqslant 3$ there is an infinite set of circulant tournaments T such that $dc(T) = r$ and $\overrightarrow{\omega}_3(T) = 3$.*

Proof. By Theorem 3 and Lemma 20, we can construct the infinite sets $\mathcal{F}_{r,3}$ for $r = 6$ and $r \geqslant 8$.

Let $\alpha \in \mathcal{F}_{2,2}$, then

$$dc\left(\overrightarrow{C}_3[\alpha]\right) = \left\lceil \frac{2(3)}{2} \right\rceil = 3 \text{ and } \overrightarrow{\omega}_3(\overrightarrow{C}_3[\alpha]) = 2 + 2 - 1 = 3.$$

Therefore $\mathcal{F}_{3,3} = \left\{\overrightarrow{C}_3[\alpha] : \alpha \in \mathcal{F}_{2,2}\right\}$.

Let $\alpha \in \mathcal{F}_{2,2}$, then

$$dc\left(\overrightarrow{C}_5\langle\emptyset\rangle[\alpha]\right) = \left\lceil \frac{2(5)}{3} \right\rceil = 4 \text{ and } \overrightarrow{\omega}_3(\overrightarrow{C}_3[\alpha]) = 2 + 2 - 1 = 3.$$

Then $\mathcal{F}_{4,3} = \left\{\overrightarrow{C}_5\langle\emptyset\rangle[\alpha] : \alpha \in \mathcal{F}_{2,2}\right\}$.

Analogously,

$$\mathcal{F}_{5,3} = \left\{\overrightarrow{C}_3[\alpha] : \alpha \in \mathcal{F}_{3,2}\right\}, \quad \mathcal{F}_{7,3} = \left\{\overrightarrow{C}_7\langle 3\rangle[\alpha] : \alpha \in \mathcal{F}_{3,2}\right\}. \qquad \square$$

Let $B(s)$ $(s \geqslant 2)$ be a positive integer such that for every $r \geqslant B(s)$ there exists an infinite set of circulant tournaments $T \in \mathcal{F}_{r,s}$. Clearly, $B(2) = 2$ and by Proposition 22, we have that $B(3) = 3$ (see Corollary 10).

Proposition 23. $B(s) \leqslant 2B(s-1) - 1 \leqslant 2^{s-2} + 1$ *for every* $s \geqslant 3$.

Proof. We proceed by induction on s. For $s = 3$, by Proposition 22, we have that

$$B(3) = b(3) = \left\lceil \frac{3}{2} \left\lceil \frac{3}{2} \right\rceil \right\rceil = 3 = 2B(2) - 1.$$

Suppose that the statement is valid for some $s \geqslant 3$. We will prove it for $s + 1$. By Proposition 21 (i) and (ii), we can construct the infinite sets

$$\begin{aligned} \mathcal{F}_{2r,s+1} &= \left\{\overrightarrow{C}_{2r+1}\langle\emptyset\rangle[\alpha] : \alpha \in \mathcal{F}_{r,s}\right\} \text{ and} \\ \mathcal{F}_{2r+1,s+1} &= \left\{\overrightarrow{C}_{2(3r+1)+1}\langle 3r\rangle[\alpha] : \alpha \in \mathcal{F}_{r,s}\right\} \end{aligned}$$

for every $r \geqslant B(s)$. It follows that $B(s+1) \leqslant 2B(s)$. We only need to prove the existence of an infinite family $\mathcal{F}_{2^{s-2}+1,s}$ for every $s \geqslant 4$.

Let $s \geqslant 3$. Notice that $2^{s-2} + 1 \equiv 0 \pmod 3$ or $2^{s-2} + 1 \equiv 2 \pmod 3$.

(i) If $2^{s-2}+1 \equiv 0 \pmod 3$, then $2^{s-2}+1 = 3q$ for some integer $q \geqslant 1$ and by Proposition 21(iii) it follows that $\mathcal{F}_{2^{s-2}+1,s} = \left\{\overrightarrow{C}_3[\alpha] : \alpha \in \mathcal{F}_{2q,s-1}\right\}$.

(ii) If $2^{s-2} + 1 \equiv 2 \pmod 3$, then $2^{s-2} + 1 = 3q + 2$ for some integer $q \geqslant 1$ and by Proposition 21(iv) it follows that $\mathcal{F}_{2^{s-2}+1,s} = \left\{\overrightarrow{C}_3[\alpha] : \alpha \in \mathcal{F}_{2q+1,s-1}\right\}$. $\square$

By Proposition 23 and Proposition 21(iii) and (iv), we can construct infinite families of circulant tournaments T with arbitrary

$$\overrightarrow{\omega}_3(T) = s \text{ and } r \geqslant \left\lceil \frac{3}{2} b(s) \right\rceil$$

for every $r \geqslant 2$, except for the cases when $r \equiv 1 \pmod 3$ and $\left\lceil \frac{3}{2} b(s) \right\rceil \leqslant r < B(s)$. For these values of r, we have to construct each infinite family case by case. Some of these families are obtained using the construction and the properties of digraph $D = D_0 \cup D_1 \cup D_2$ (see Lemmas 19 and 20). Others are obtained by the composition $\overrightarrow{C}_{2m+1}\langle m-1\rangle[\alpha]$, where α is an r-dichromatic circulant tournament with $r > m$. Since $r > m$, these families are not those $\mathcal{F}_{2r,s+1} = \{\overrightarrow{C}_{2m+1}\langle\emptyset\rangle[\alpha] : \alpha \in \mathcal{F}_{r,s}\}$ given in Proposition 21(i). Working with the aforementioned tools, one could obtained the following infinite sets $\mathcal{F}_{r,s}$ of circulant tournaments:

(i) $3 \leqslant s \leqslant 5$ and every $r \geqslant b(s)$,

(ii) $s = 6$ and every $r \geqslant 12$ and $r \neq 13$,

(iii) $s = 7$ and $r = 18, 21$ and every $r \geqslant 25$,

(iv) $s = 8$ and $r = 27, 32$ and every $r \geqslant 35$,

(v) $s = 9$ and $r = 41, 48, 51$ and every $r \geqslant 53$,

(vi) $s = 10$ and $r = 62, 72, 76, 77$ and every $r \geqslant 80$ and

(vii) $s = 11$ and $r = 93, 108, 114, 121$ and every $r \geqslant 123$.

Cases (i) and (ii) are consequences of Proposition 22, Theorem 3 and using that $b(s) = B(s)$ for $s \in \{2,3,4\}$, $b(6) = 12$ and $B(6) = 14$. Observe that $\mathcal{F}_{13,6}$ is the first unknown infinite set.

In Case (iii), we have that $r \geqslant 18$ by Theorem 3. By Proposition 23, there is an infinite set $\mathcal{F}_{r,7}$ for every $r \geqslant 25$. For $r = 21$, let $\alpha \in \mathcal{F}_{14,6}$. Hence $dc\left(\overrightarrow{C}_3[\alpha]\right) = \left\lceil \frac{14(3)}{2} \right\rceil = 21$ and $\overrightarrow{\omega}_3\left(\overrightarrow{C}_3[\alpha]\right) = 2 + 6 - 1 = 7$ by Corollary 18 and Proposition 12, respectively. Therefore, $\mathcal{F}_{21,7} = \left\{\overrightarrow{C}_3[\alpha] : \alpha \in \mathcal{F}_{14,6}\right\}$. For $r \in \{19, 20, 22, 23, 24\}$, the infinite sets $\mathcal{F}_{r,7}$ are unknown. Let $r = 25$ and $\alpha \in \mathcal{F}_{12,6}$. Then $dc\left(\overrightarrow{C}_{75}\langle 36\rangle[\alpha]\right) = \left\lceil \frac{12(75)}{36} \right\rceil = 25$ (Corollary 18 for $m = 37$) and $\overrightarrow{\omega}_3\left(\overrightarrow{C}_{75}\langle 36\rangle[\alpha]\right) = 2 + 6 - 1 = 7$ (Proposition 12). For $r = 26$, take $\alpha \in \mathcal{F}_{17,6}$ and apply Proposition 21(iv) to obtain $\mathcal{F}_{26,7} = \left\{\overrightarrow{C}_3[\alpha] : \alpha \in \mathcal{F}_{17,6}\right\}$. For $r = 27$, take $\alpha \in \mathcal{F}_{18,6}$ and apply Proposition 21(iii) and we get $\mathcal{F}_{27,7} = \left\{\overrightarrow{C}_3[\alpha] : \alpha \in \mathcal{F}_{18,6}\right\}$. Finally, note that $B(7) \leqslant 2B(6) - 1 = 2 \cdot 14 - 1 = 27$ (see Proposition 23).

Using similar arguments, one can obtain the remaining cases.

The following table summarizes some exact values of $b(s)$ and $B(s)$.

s	2	3	4	5	6	7	8	9	10	11
$b(s)$	2	3	5	8	12	18	27	41	62	93
$B(s)$	2	3	5	9	14	25	35	53	80	123

We finish with the following conjecture.

Conjecture 24. *$B(s) \leqslant 2b(s-1)-1$ for every $s \geqslant 3$.*

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