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Controlling Chaos for a Fractional-Order Discrete System

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ABSTRACT In this article two different controllers for the stabilization of a fractional-order discrete system in the left Caputo discrete delta operator sense are given. The first one acts by a fractional proportional pulse control, the second acts by a fractional feedback control. These controllers are applied to fractional-order chaotic discrete dynamical systems to obtain their stability and also we show a comparison with the integer order dynamical systems stability. Some simulations are presented for the fractional logistic map and the fractional Henon map.

INDEX TERMS Caputo delta difference, chaos, discrete fractional calculus, Mittag–Leffler.

I. INTRODUCTION

In THE last decades, a considerable number of investigations has been devoted to the analysis of chaos control. The control of chaos is understood by the stabilization of discrete or continuous chaotic systems to a steady state or a periodic orbit.

In the case of a dynamic system it is understood that control is the manipulation of different variables and parameters of the system by means of a law that is a rule or algorithm such that, when applied, a desired behaviour is obtained. In chaotic systems, manoeuvres to control chaos have been tested in different branches such as laboratory physics, cardiology and biochemistry.

Some researches present a way to reduce chaos and even eliminate it completely. The particular case Ott-Grebogi-Yorke method has been used in the Duffing oscillator using periodic proportional pulses allowing to obtain periodic orbits of period 1 to 4 [1], [2], [3], [4], [5].

In recent decades, chaotic systems and their applications have attracted attention of the scientific community in several fields of sciences and engineering. However in recent years chaotic systems of fractional order and their applications have established that some models provide better results on specific phenomena. Moreover when a controller is provided, these results could enhance [6].

There are representations of linear and non-linear systems using fractional calculus in the continuous or discrete state space domain. Through the discretization of continuous fractional integrals and derivatives, fractional difference equations have lead to a new area of research.

The fractional logistic difference equation and the Henon map using the Caputo delta operator have been studied. Unlike interger-order models, fractional ones present a discrete memory and they depend on the parameter ν, the fractional difference order. It is interesting to note in the case of the fractional logistic map that the presence of chaotic behavior depend not only on the natural parameter but also on the order ν [7], [8], [9], [10].

Stability theory and synchronization have been developed to analyse the behaviour of solutions, in particular Lyapunov functions let us identify some features of solutions like monotonicity and asymptotic behaviour. Suitable Lyapunov functions allow us to know about monotonous and stable solutions and also asymptotic ones (Lyapunov stability) [11], [12], [13].

Chaos synchronization, a master and a slave system have been proposed to establish the existence and stability of solutions corresponding to the movement of the system [14], [15], [16]. Stability analysis using Lyapunov functions and synchronization has been applied not only in the logistic fractional discrete equation, but also in the well-known two-dimensional Henon map [17].

Some investigations have used control methods to chaos fractional discrete maps [18], [19]. In [18] used this method of momentum to control chaotic dynamics of logistic fractional map. On another hand, in [19] a kicked method is applied on a damped rotator map. An additional way to control fractional maps is through a fractional linearization method, a one-dimensional controller to stabilize the system [20], [21], [22], [23].

Unlike the works mentioned above, our proposed method is based on the negative definitiveness of a matrix both for the case of proportional pulses and for the case of feedback, which allows us to guarantee stability in the Mittag-Leffler sense, which implies asymptotic stability. A comparison is also made between the integer and fractional order controller.

This article is organized as follows. The necessary definitions and theorems are presented in the second section. The third section describes the fractional order discrete systems. Two types of control are also shown, the fractional proportional pulse control and the fractional feedback control. The fourth section shows some applications of these controls for the fractional logistic model and the fractional Henon map. Finally, conclusions are given.

II. PRELIMINARIES

In this section we start with some basic definitions about discrete fractional calculus, necessary to construct the controller for fractional discrete systems. For $a \in \mathbb{R}$, the functions we deal with are defined in the Banach space $S(\mathbb{N}_a, \mathbb{R})$, the set of all functions $f : \mathbb{N}_a \to \mathbb{R}$, whose domain is the discrete set $\mathbb{N}_a := \{a, a+1, a+2, \ldots\}$. We introduce the forward operator $\sigma : \mathbb{N}_a \to \mathbb{N}_a$ as $\sigma(t) := t+1$ and the corresponding forward difference operator by $\Delta f(t) = f(\sigma(t)) - f(t)$. We get

$$\Delta^m f(t) = \Delta\Big(\Delta^{m-1} f(t)\Big) = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} f(t+k).$$

We use the notation $S_\rho^{n-1} = \{\mathbf{x} \in \mathbb{R} | \|\mathbf{x}\| \le \rho\}$. The following definitions were recovered from [24].

Definition 1: Let $u \in S(\mathbb{N}_a, \mathbb{R})$ and $v > 0$ be given, the v-fractional sum of u is defined by

$$\Delta_a^{-v} u(t) = \frac{1}{\Gamma(v)} \sum_{s=a}^{t-v} (t - \sigma(s))^{(v-1)} u(s) \tag{1}$$

where a is the starting point, Γ is the function gamma, as an extension of the factorial function, and

$$t^{(v)} = \frac{\Gamma(t+1)}{\Gamma(t+1-v)} \tag{2}$$

corresponds to the falling function. From this definition, $\Delta_a^{-v} : S(\mathbb{N}_a, \mathbb{R}) \to S(\mathbb{N}_{a+v}, \mathbb{R})$.

In what follows, the fractional difference we assume is that proposed by Caputo.

Definition 2: The v-order Caputo discrete fractional difference for $v > 0$ ($v \notin \mathbb{N}$) and $u(t) \in S(\mathbb{N}_a, \mathbb{R})$ is given by

$${}^C\Delta_a^v u(t) = \Delta^{-(m-v)}\big(\Delta^m u(t)\big) \tag{3}$$

where ${}^C\Delta_a^v : S(\mathbb{N}_a, \mathbb{R}) \to S(\mathbb{N}_{a+(m-v)}, \mathbb{R})$ is the Caputo delta operator, with $m-1 < v < m$. That is

$${}^C\Delta_a^v u(t) = \frac{1}{\Gamma(m-v)} \sum_{s=a}^{t-(m-v)} (t - \sigma(s))^{(m-v-1)} \Delta_s^m u(s) \tag{4}$$

where $t \in \mathbb{N}_{a+m-v}$.

From [24] we have a Taylor's difference formula for the Caputo discrete fractional difference.

Theorem 1: For $v > 0$, v non-integer, it holds:

$$f(t) = \sum_{k=0}^{m-1} \frac{(t-a)^{(k)}}{k!} \Delta^k f(a) + \frac{1}{\Gamma(v)} \sum_{s=a+m-v}^{t-v} (t-\sigma(s))^{(v-1)}\, {}^C\Delta_a^v f(s). \tag{5}$$

In particular for $0 < v < 1$ we know

$$f(t) = f(a) + \frac{1}{\Gamma(v)} \sum_{s=a+1-v}^{t-v} (t - \sigma(s))^{(v-1)}\, {}^C\Delta_a^v f(s). \tag{6}$$

It lead us to a solution for the nonlinear fractional Caputo like difference equation initial value problem

$${}^C\Delta_o^v u(t) = f(u(t+v-1), t+v-1), \quad u(0) = u_o, \tag{7}$$

for $t \in \mathbb{N}_{1-v}$, whenever $v \in (0,1)$, which corresponds to

$$u(t) = u_o + \frac{1}{\Gamma(v)} \sum_{s=1-v}^{t-v} (t-s-1)^{(v-1)} \times f(u(s+v-1), s+v-1) \tag{8}$$

with $u(0) = u_o$.

Definition 3: A fixed point $\mathbf{u} = 0$ of equation (7) is said to be

- Stable if for every $\epsilon > 0$ and $t_0 \in \mathbb{N}_{1-v}$ there exists $\delta_{\epsilon,t_0} > 0$ such that every solution $\mathbf{u}(t)$ with initial condition satisfying $\|\mathbf{u}_0\| < \delta_{\epsilon,t_0}$ implies $\|\mathbf{u}(t)\| < \epsilon$, for all $t \in \mathbb{N}_{t_0}$,
- Uniformly stable if it is stable and δ just depend on ϵ,
- Asymptotically stable if it is stable and for all $t_0 \in \mathbb{N}_{1-v}$ there exist $\delta_{t_0} > 0$ such that $lim_{t\to\infty}\mathbf{u}(t) = 0$ whenever $\|\mathbf{u}_0\| < \delta_{t_0}$.

Definition 4: For $\alpha, \beta, z \in \mathbb{C}$, with $Re(\alpha) > 0$, the discrete Mittag-Leffler two-parameter function is given as

$$\mathscr{E}_{\alpha,\beta}(\lambda, z) := \sum_{k=0}^{\infty} \lambda^k \frac{(z+(k-1)(\alpha-1))^{(k\alpha)} (z + k(\alpha-1))^{(\beta-1)}}{\Gamma(\alpha k + \beta)}$$

where $E_{(\alpha)}(\lambda, z) := \mathscr{E}_{\alpha,1}(\lambda, z)$.

Definition 5: A solution to (7) is said to be stable Mittag-Leffler if

$$\|\mathbf{u}(t)\| \le \big(m(\mathbf{u}_0) E_{(\alpha)}(-\lambda, t)\big)^b$$

where $\alpha \in (0,1)$, $\lambda > 0$, $b > 0$, $m(\mathbf{0}) = 0$, $m(\mathbf{u}) \ge 0$, and m locally Lipschitz on $\mathbf{u} \in \mathbb{B} \subset \mathbb{R}^n$ with Lipschitz constant m_0.

III. FRACTIONAL-ORDER DISCRETE SYSTEMS

Let us consider the autonomous version of an n-dimensional nonlinear function

$$ {}^{C}\Delta_{o}^{v}\mathbf{x}(t) = \mathbf{f}(\mathbf{x}(t+v-1)), \mathbf{x}(0) = \mathbf{x}_0 \quad (9)$$

where ${}^{C}\Delta_{o}^{v}x(t)$ is the Caputo-like delta difference and so we can compute

$$\mathbf{x(t)} = \mathbf{x}_0 + \frac{1}{\Gamma(v)}\sum_{s=0}^{t-1}\frac{\Gamma(t-s+v-1)}{\Gamma(t-s)}\mathbf{f}(\mathbf{x}(s)) \quad (10)$$

We assume that the eq. (10) can be rewritted as follows

$$ {}^{C}\Delta_{o}^{v}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t+v-1) + \mathbf{g}(\mathbf{x}(t+v-1)), \quad (11)$$

a constant matrix $\mathbf{A}$ standing for the linear part and the function $\mathbf{g}$ for the nonlinear terms of ${}^{C}\Delta_{o}^{v}\mathbf{x}(t)$.

A. A FRACTIONAL PROPORTIONAL PULSE CONTROL

A specific method to control chaos in integer-order systems has been introduced by [5]. In the case of fractional discrete systems, can be applied a proportional pulse control in (11)

$$ {}^{C}\Delta_{o}^{v}\mathbf{x}(t) = \mathbf{K}[\mathbf{A}\mathbf{x}(t+v-1)] + \mathbf{g}(\mathbf{x}(t+v-1)), \quad (12)$$

where $\mathbf{K}$ is a constant matrix. The linear part of (12) is given by the term $\mathbf{KA}[\mathbf{x}(t+v-1)]$. The associated linear system is Mittag-Leffler stable by [24, Th. 4] if the matrix $\mathbf{KA}$ is negative definite.

The results of the following Theorems 2 and 3 are based on the [24, Th. 6].

Theorem 2: Suppose that $\mathbf{x} = \mathbf{0}$ is a fixed point of (9) and the function $\mathbf{f} : S_r^{n-1} \times \mathbb{N}_{v-1} \to \mathbb{R}^n$ locally Lipschitz for $\mathbf{x} \in S_r^{n-1}$ such that the dimension of the proper linear eigenspace satisfies $\dim E_{\sigma_A} = n$, where the spectrum of $\mathbf{A}$ is $\sigma_A = \lambda_i$ with at least one eigenvalue nonzero, then there exists a constant matrix $\mathbf{K}$ such that (9) is Mittag-Leffler stable.

Proof:

- If $\mathbf{A}$ and $\mathbf{K}$ have real eigenvalues. Consider a matrix $\mathbf{A}_{n\times n}$ and its decomposition $\mathbf{A} = \mathbf{P}\mathbf{J_A}\mathbf{P}^{-1}$, where $\mathbf{J_A}$ is the Jordan canonical form, with $\sigma_{\mathbf{A}} = \{\lambda_i\}$ the eigenvalues. We propose $\mathbf{K}$ with a similar decomposition such that $\mathbf{K} = \mathbf{P}\mathbf{J_K}\mathbf{P}^{-1}$. In this way, the product $\mathbf{KA}$ lead us to $\mathbf{P}\mathbf{J_K}\mathbf{J_A}\mathbf{P}^{-1}$. That is $\sigma_{\mathbf{KA}} = \mu_i\lambda_i$ where $\sigma_{\mathbf{K}} = \{\mu_i\}$. Therefore we define $\mathbf{K}$ in such a way that sgn $\mu_i = -$ sgn λ_i.
- If $\mathbf{A}$ and $\mathbf{K}$ have complex eigenvalues. Let $\mathbf{A}_{n\times n}$ be a real matrix with complex eigenvalues and its decomposition $\mathbf{A}= \mathbf{P}\mathbf{J_A}\mathbf{P}^{-1}$ where $\mathbf{J_A}$ is a Jordan canonical form. Note $\mathbf{J_A}$ is a diagonal matrix by blocks $\mathbf{J_A} = diag(J_1(\lambda_1), J_2(\lambda_2), \dots, J_n(\lambda_s))$ and $J_i(\lambda_j) = diag(\lambda_1, \lambda_2, \dots, \lambda_s)$, where s represents the number of eigenvalues of the matrix taking into account multiplicity. Taking $\mathbf{K}$ a matrix which decomposition is $\mathbf{K}= \mathbf{P}\mathbf{J_K}\mathbf{P}^{-1}$ now $\mathbf{J_K}$ is a diagonal matrix by blocks with $\mathbf{J_K} = diag(J_1(\mu_1), J_2(\mu_2), \dots, J_n(\mu_s))$ and $J_i(\mu_j) = diag(\mu_1, \mu_2, \dots, \mu_s)$ where s again represents the number of eigenvalues of the matrix taking into account their multiplicity, that is

$$\mathbf{J_A} = \begin{pmatrix} a_1 & b_1 & \cdots & 0 & 0 \\ -b_1 & a_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_n & b_n \\ 0 & 0 & \cdots & -b_n & a_n \end{pmatrix}$$

$$\mathbf{J_K} = \begin{pmatrix} \alpha_1 & \beta_1 & \cdots & 0 & 0 \\ -\beta_1 & \alpha_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \alpha_n & \beta_n \\ 0 & 0 & \cdots & -\beta_n & \alpha_n \end{pmatrix}$$

Taking the product of matrices $\mathbf{J_K}\mathbf{J_A}$, is obtained

$$\mathbf{J_K}\mathbf{J_A} = \begin{pmatrix} J_1 & 0 & \cdots & 0 & 0 \\ 0 & J_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & J_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & J_n \end{pmatrix}$$

where $J_1, J_2, \dots, J_n$ are block matrices. The matrix J_i given by

$$J_i = \begin{pmatrix} \alpha_i a_i - \beta_i b_i & \beta_i a_i + \alpha_i b_i \\ -\beta_i a_i - \alpha_i b_i & \alpha_i a_i - \beta_i b_i \end{pmatrix}$$

Let sgn $(\beta_i) =$ sgn (b_i) and $\alpha_i = -(a_i/b_i)\beta_i$, therefore

$$J_i = \begin{pmatrix} \alpha_i a_i - \beta_i b_i & 0 \\ 0 & \alpha_i a_i - \beta_i b_i \end{pmatrix}$$

$$J_i = c\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

with $c = \alpha_i a_i - \beta_i b_i < 0$ for each $i = 1, 2, \dots, n$. So, by [24, Th. 6] the fixed point is Mittag-Leffler stable. ■

B. FRACTIONAL FEEDBACK CONTROL

In this subsection we show a new fractional controller method. From the system (11) and just tackling the linear part let us introduce the following modified linear system

$$ {}^{C}\Delta_{o}^{v}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t+v-1) + \mathbf{B}\mathbf{u}(t+v-1) \quad (13)$$

where $\mathbf{u}(t+v-1) = -\mathbf{K}\mathbf{x}(t+v-1)$ and $\mathbf{A}_{n\times n}$, $\mathbf{B}_{n\times m}$, $\mathbf{K}_{m\times n}$ are constant matrices. Substituting in (13) we have

$$ {}^{C}\Delta_{o}^{v}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t+v-1) - \mathbf{B}\mathbf{K}\mathbf{x}(t+v-1) \quad (14)$$

$$ {}^{C}\Delta_{o}^{v}\mathbf{x}(t) = (\mathbf{A} - \mathbf{BK})\mathbf{x}(t+v-1) \quad (15)$$

Stability is now determined by the eigenvalues of $(\mathbf{A} - \mathbf{BK})$.

Theorem 3: Suppose $\mathbf{x} = \mathbf{0}$ is a fixed point of (9) and the function $\mathbf{f} : S_r^{n-1} \times \mathbb{N}_{v-1} \to \mathbb{R}^n$ locally Lipschitz for $\mathbf{x} \in S_r^{n-1}$. Therefore $\mathbf{x} = \mathbf{0}$ is Mittag-Leffler stable if $(\mathbf{A}-\mathbf{BK})$ is negative definite.

The proof of Theorem 3 is similar to the proof of Theorem 2, but using the [24, Th. 6]. In this case we have to choose one of following type of control.

Let $(\mathbf{A}-\mathbf{BK})=\mathbf{M}$ such that $\mathbf{x}^{\mathbf{T}}\mathbf{Mx}<0$,

- Case 1) We suppose $\mathbf{B}^{-1}$ exist. In this case $\mathbf{A},\mathbf{B}$ are given and determined by the system and the controller, so we can choice $\mathbf{K}=\mathbf{B}^{-1}(\mathbf{A}-\mathbf{M})$.
- Case 2) Now suppose $\mathbf{B}$ is invertible for the right $Rank(\mathbf{B}_{m\times n})=m$, i.e., $\mathbf{BB}^{d}=\mathbf{I}$ but $\mathbf{B}^{d}\mathbf{B}\neq\mathbf{I}$. For this case, also $\mathbf{A},\mathbf{B}$ are given and determinate by the system and the controller, so we choose $\mathbf{K}=\mathbf{B}^{d}(\mathbf{A}-\mathbf{M})$.
- Case 3) We assume that the generalized inverse of $\mathbf{B}$ denoted by $\mathbf{B}^{+}$ is such that $\mathbf{BB}^{+}=\mathbf{I}$, then we select the controller $\mathbf{K}=\mathbf{B}^{+}(\mathbf{A}-\mathbf{M})$.

Remark that controllers 2) and 3) are less conservative, since they only assume the existence of a right inverse of matrix $\mathbf{B}$.

IV. SOME APPLICATIONS

Below we show applications of the above results to stabilize this class of discrete fractional systems.

A. FRACTIONAL ORDER LOGISTIC MAP

In this first subsection, we will describe the fractional order logistic map. The classical logistic map model deeply studied is defined as

$$x(n+1)=\mu x(n)(1-x(n)),\, x(0)=c \tag{16}$$

where the model exhibits chaotic behaviour for values of $\mu\in(3.57,4]$. The fractional logistic map of order v can be expressed as

$$\begin{aligned}{}^{C}\Delta_{a}^{v}x(t)&=\mu x(t+v-1)(1-x(t+v-1))\\&\quad-x(t+v-1),\end{aligned} \tag{17}$$

Therefore, the solution of the equation (17) is given by

$$\begin{aligned}x(t)&=x(0)+\frac{\mu}{\Gamma(v)}\sum_{s=a+m-v}^{t-v}(t-\sigma(s))^{(v-1)}\\&\quad\times f(s+v-1,x(s+v-1))\end{aligned} \tag{18}$$

with $t\in\mathbb{N}$ and the classical case is obtained for $m=1$. A numerical solution of the equation (18) can be written as,

$$\begin{aligned}x(t)&=x(0)+\frac{\mu}{\Gamma(v)}\sum_{j=1}^{t}\frac{\Gamma(t-j+v)}{\Gamma(t-j+1)}\\&\quad\times\left[x(j-1)(1-x(j-1))-x(j-1)\right]\end{aligned} \tag{19}$$

Taking the linear part

$$Ax(t+v-1)=\mu x(t+v-1)-x(t+v-1), \tag{20}$$

In order to apply the proportional pulse, we mutiply by K

$$KAx(t+v-1)=K\mu x(t+v-1)-Kx(t+v-1) \tag{21}$$

From Theorem 2, we compute the derivate, $K(\mu-1)$. Therefore if $K>0$ then $(\mu-1)<0$ for $\mu<1$. For the opposite case, if $K<0$ then $(\mu-1)>0$ for $\mu>1$.

Figure 1 shows chaotic dynamics of the fractional order logistic model. Each trajectory obey a fractional order v, with the fixed value $\mu=2.5$. Figures 2, 4, 6 show dynamics under variations in v, for a fixed value μ.

In Figures 3, 5 and 7 is depicted the fractional proportional pulse control for these model. Simulations show a consistent behaviour for values of $K<0$: for small negative values we observe the convergence towards the stable state is smoother and slower and for negative values close to zero observe the convergence is faster.

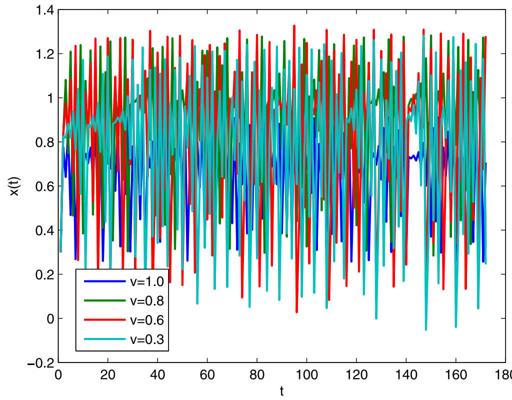

FIGURE 1. Time series for the Caputo Fractional Logistic Map with $\mu=2.5$ and $v=1.0, 0.8, 0.6, 0.3$.

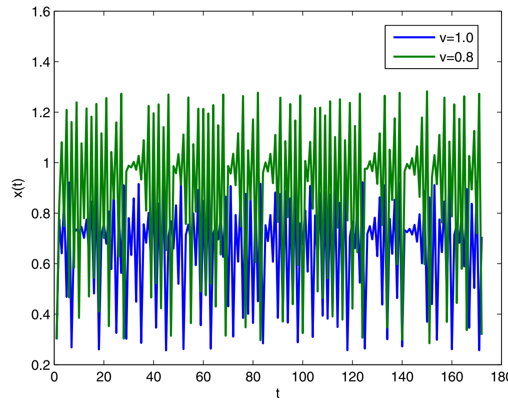

FIGURE 2. Time Series of the Fractional Logistic Map with $\mu=2.5$, $v=1.0, 0.8$.

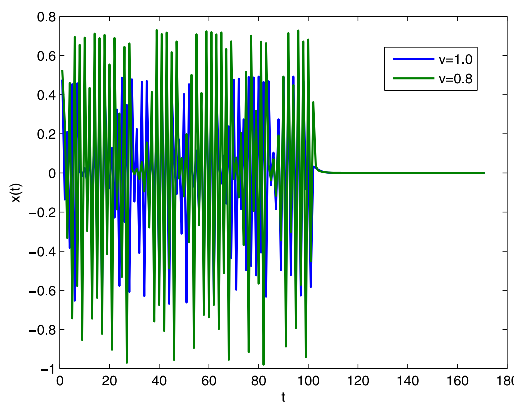

FIGURE 3. Proportional Pulse Control in Fractional Order Logistic Map with $\mu=2.5$, $v=1.0, 0.8$ and $K=-0.6$.

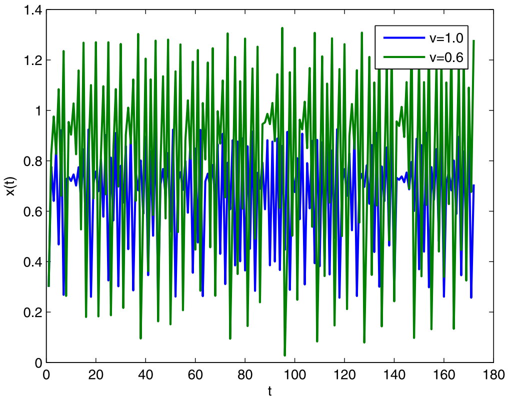

FIGURE 4. Time Series of the Fractional Logistic Map with $\mu = 2.5$, $v = 1.0, 0.6$.

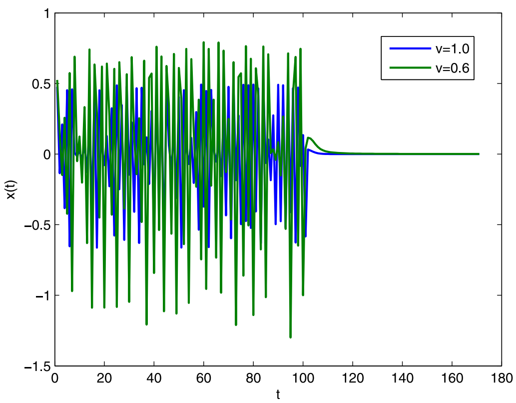

FIGURE 5. Proportional Pulse Control in Fractional Order Logistic Map with $\mu = 2.5$, $v = 0.6$ and $K = -0.3$.

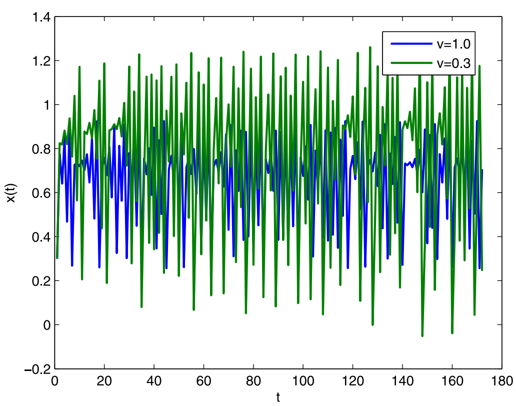

FIGURE 6. Time Series of the Fractional Logistic Map with $\mu = 2.5$, $v = 1.0, 0.3$.

It should be noted that in all cases, the stabilization of the system is obtained regardless the value of v and the values of μ for which chaotic dynamics take place; we choose a K that stabilizes the model. Note that the oscillations in the fractional logistic map are greater than the oscillations in the integer logistic map.

B. FRACTIONAL ORDER HENON MAP

Now we tackle another classical map. This integer model is a quadratic two-dimensional system whose parameters α

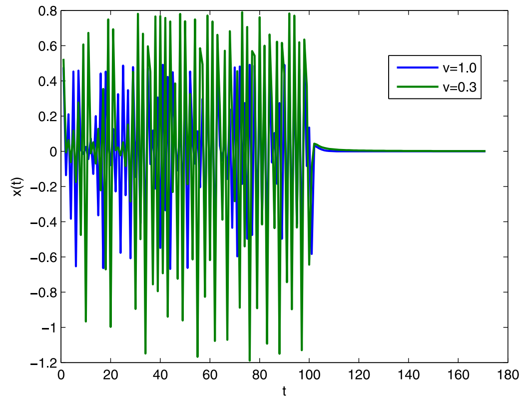

FIGURE 7. Proportional Pulse Control in Fractional Order Logistic Map with $\mu = 2.5$, $v = 0.3$ and $K = -0.1$.

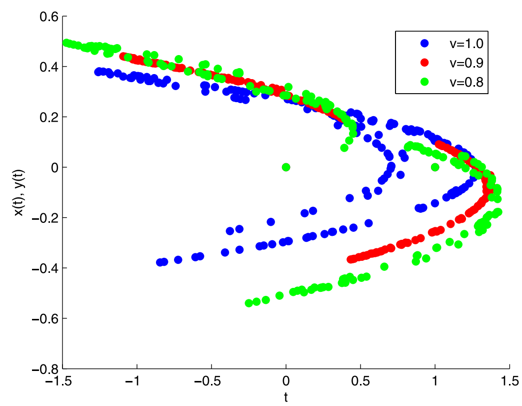

FIGURE 8. Fractional and Integer Order Henon Map with $\alpha = 1.4$, $\beta = 0.3$ and $v = 1.0, 0.9, 0.8$.

and β are positive.

$$\begin{aligned} x(n+1) &= y(n) + 1 - \alpha x^2(n) \\ y(n+1) &= \beta x(n) \end{aligned} \tag{22}$$

The fractional Henon map of fractional order v, can be expressed as

$$\begin{aligned} {}^C\Delta_a^v x(t) &= y(t+v-1) + 1 - \alpha x^2(t+v-1) - x(t+v-1) \\ {}^C\Delta_a^v y(t) &= \beta x(t+v-1) - y(t+v-1) \end{aligned} \tag{23}$$

where fixed points (x_1, y_1) and (x_2, y_2) are given by

$$\left(\frac{1-\beta+\sqrt{(\beta-1)^2+4\alpha}}{-2\alpha}, \frac{\beta-\beta^2+\beta\sqrt{(\beta-1)^2+4\alpha}}{-2\alpha} \right)$$

$$\left(\frac{1-\beta-\sqrt{(\beta-1)^2+4\alpha}}{-2\alpha}, \frac{\beta-\beta^2-\beta\sqrt{(\beta-1)^2+4\alpha}}{-2\alpha} \right)$$

respectively. The matrix $\mathbf{A}$ standing for the linear part at each fixed point is

$$\begin{pmatrix} -2\alpha x_{1,2} - 1 & 1 \\ \beta & -1 \end{pmatrix} = \begin{pmatrix} -\beta \pm \sqrt{(\beta-1)^2+4\alpha} & 1 \\ \beta & -1 \end{pmatrix}$$

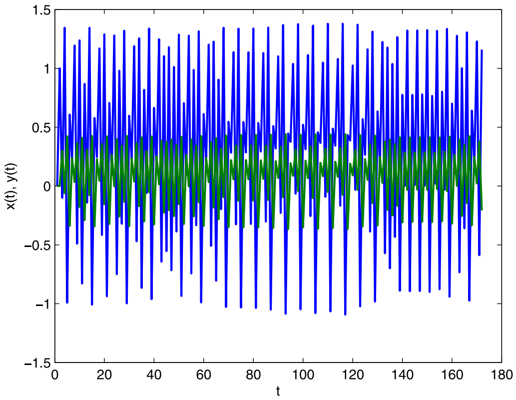

FIGURE 9. Time Series of Fractional Order Henon Map with $\alpha = 1.4$, $\beta = 0.3$ and $v = 0.9$.

A numerical solution of the previous system can be written as,

$$x(t) = x(0) + \frac{1}{\Gamma(v)} \sum_{j=1}^{t} \frac{\Gamma(t-j+v)}{\Gamma(t-j+1)} \times \left[y(j-1) + 1 - \alpha x^2(j-1) - x(j-1)\right]$$
$$y(t) = y(0) + \frac{1}{\Gamma(v)} \sum_{j=1}^{t} \frac{\Gamma(t-j+v)}{\Gamma(t-j+1)} \times \left[\beta x(j-1) - y(j-1)\right] \tag{24}$$

From the equation (23) taking the linear part and adding a linear perturbation via the matrix $\mathbf{B} = \mathbf{I}$ we written the system

$$^{C}\Delta_a^v \mathbf{x}(t+v-1) = (\mathbf{A} - \mathbf{BK})\mathbf{x}(t+v-1) \tag{25}$$

with

$$K = \begin{pmatrix} k_1 & 1 \\ \beta & k_2 \end{pmatrix}$$
$$\begin{pmatrix} ^{C}\Delta_a^v x(t) \\ ^{C}\Delta_a^v y(t) \end{pmatrix} = \left[\begin{pmatrix} -\beta - k_1 \pm \sqrt{(\beta-1)^2 + 4\alpha} & 1 \\ \beta & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k_1 & 1 \\ \beta & k_2 \end{pmatrix} \right] \begin{pmatrix} x(t+v-1) \\ y(t+v-1) \end{pmatrix}$$

Defining $\mathbf{M} = (\mathbf{A} - \mathbf{BK})$ at the fixed point (x_2, y_2), we know

$$\mathbf{M} = \begin{pmatrix} -\beta - k_1 - \sqrt{(\beta-1)^2 + 4\alpha} & 0 \\ 0 & -1 - k_2 \end{pmatrix} \prec 0$$

when $k_1 > -\beta - \sqrt{(\beta-1)^2 + 4\alpha}$ and $k_2 > -1$ from the computation

$$\begin{bmatrix} a & b \end{bmatrix} \mathbf{M} \begin{bmatrix} a \\ b \end{bmatrix} = a^2 \left(-(\beta + k_1) \pm \sqrt{(\beta-1)^2 + 4\alpha} \right) - b^2(1 + k_2) < 0 \tag{26}$$

for $(a, b) \in \mathbb{R}^2$.

The Henon map of fractional and integer order is presented in Figure 8. These plots compares the dynamics for $x(t)$, $y(t)$ solutions of integer and fractional order. In Figures 9 and 10 are shown time series for $x(t)$ and $y(t)$ with the parameters of $\alpha = 1.4$ and $\beta = 0.3$ for a fractional order $v = 0.9$ and $v = 0.8$.

Given Theorem 3 and the equation (26), the Henon map is stabilized. Figures 11 and 12 describe the time series of fractional Henon map when applied the feedback control and we realize the stabilization of solutions.

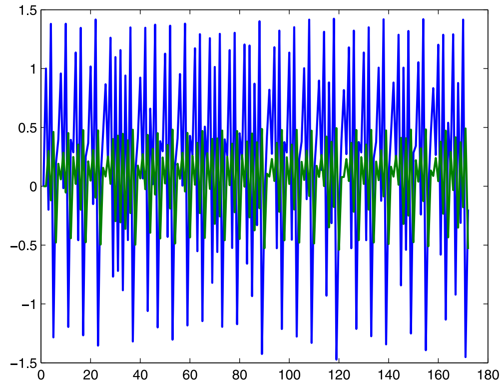

FIGURE 10. Time Series of Fractional Order Henon Map with $\alpha = 1.4$, $\beta = 0.3$ and $v = 0.8$.

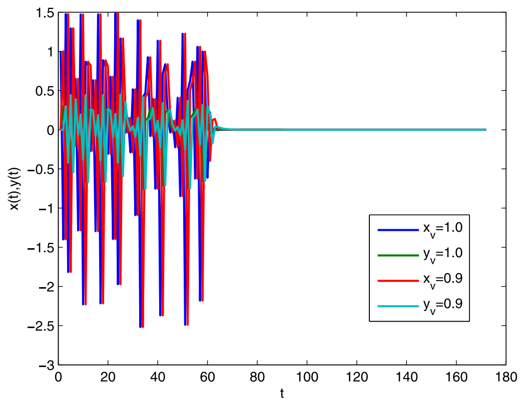

FIGURE 11. Feedback Control in the Fractional Henon Map with $\alpha = 1.4$, $\beta = 0.3$, $v = 1.0, 0.9$ and $k_{1,2} = 0.4$.

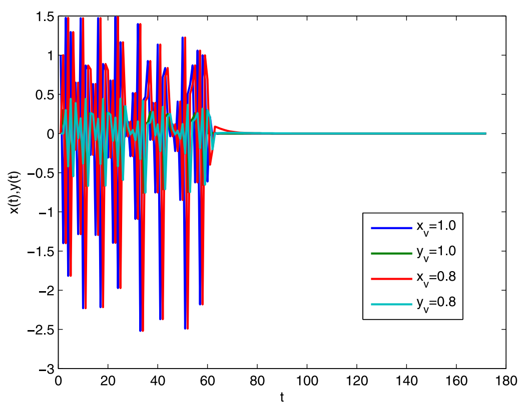

FIGURE 12. Difference for Fractional Order Henon Map, with $\alpha = 1.4$, $\beta = 0.3$, $v = 1.0, 0.8$, $k_1 = 0.3$ and $k_2 = 0.1$.

Figures 11 and 12 show the dynamics after applying the feedback control; fixing $k_1, k_2 = 0, 4$ and $k_1 = 0.1, k_2 = 0.3$ stabilization is reached, the difference between them is how fast is reached the stable state.

As the same case of fractional logistic map, the oscillations of fractional Henon map are greater than the oscillations in the integer Henon map.

V. CONCLUSION

In this article, fractional order discrete systems in the Caputo sense are presented, where chaotic behavior is described. In order to stabilize chaotic dynamics, two types of controls are proposed: a fractional proportional pulse control and a fractional feedback control. These controllers act in the linear part on the fractional nonlinear system in order to stablish Mittag-Leffler stability.

A comparison applying control was made between the fractional and integer order equations. The two control methods we present are very simple in the implementation way and Mittag-Leffler type stability is obtained, but they are more conservative. Finally these proposed controls are used to stabilize the chaotic dynamics of the fractional logistics model and the Henon fractional map.

As future work we expect to study control sensitivity, robustness, control effort and their characteristics of both controllers proposed.

REFERENCES

[1] Y. Braiman and I. Goldhirsch, "Taming chaotic dynamics with weak periodic perturbations," *Phys. Rev. Lett.*, vol. 66, no. 20, pp. 2545–2548, 1991.
[2] U. Dressler and G. Nitsche, "Controlling chaos using time delay coordinates," *Phys. Rev. Lett.*, vol. 68, no. 1, pp. 1–5, 1992.
[3] J. Güémez and M. A. Matias, "Control of chaos in unidimensional maps," *Phys. Lett. A*, vol. 181, no. 1, pp. 29–32, 1993.
[4] M. A. Matias and J. Güémez, "Stabilization of chaos by proportional pulses in the system variables," *Phys. Rev. Lett.*, vol. 72, no. 10, pp. 1455–1458, 1994.
[5] N. P. Chau, "Controlling chaos by periodic proportional pulses," *Phys. Lett. A*, vol. 234, no. 3, pp. 193–197, 1997.
[6] L. A. Quezada-Téllez and L. Franco-Pérez, "A fractional logistic approach for economic growth," *Int. J. Mod. Phys. C*, vol. 29, no. 12, 2018, Art. no. 1850123.
[7] G. C. Wu and D. Baleanu, "Discrete fractional logistic map and its chaos," *Nonlinear Dyn.*, vol. 75, nos. 1–2, pp. 283–287, 2014.
[8] G. C. Wu, D. Baleanu, H. P. Xie, and S. D. Zeng, "Discrete fractional diffusion equation of chaotic order," *Int. J. Bifurcation Chaos*, vol. 26, no. 1, 2016, Art. no. 1650013.
[9] Y. Ji, L. Lai, S. Zhong, and L. Zhang, "Bifurcation and chaos of a new discrete fractional-order logistic map," *Commun. Nonlinear Sci. Numer. Simulat.*, vol. 57, pp. 352–358, May 2018.
[10] A. Deshpande and V. Daftardar-Gejji, "Chaos in discrete fractional difference equations," *Pramana*, vol. 87, no. 4, pp. 1–10, 2016.
[11] F. Jarad, T. Abdeljawad, D. Baleanu, and K. Bicen, "On the stability of some discrete fractional nonautonomous systems," *Abstract Appl. Anal.*, vol. 2012, Jan. 2012, Art. no. 476581.
[12] D. Baleanu, G. C. Wu, Y. R. Bai, and F. L. Chen, "Stability analysis of Caputo-like discrete fractional systems," *Commun. Nonlinear Sci. Numer. Simulat.*, vol. 48, pp. 520–530, Sep. 2017.
[13] Y. Wei, Y. Chen, T. Liu, and Y. Wang, "Lyapunov functions for nabla discrete fractional order systems," *ISA Trans.*, vol. 88, pp. 82–90, May 2019.
[14] B. Xin, L. Liu, G. Hou, and Y. Ma, "Chaos synchronization of nonlinear fractional discrete dynamical systems via linear control," *Entropy*, vol. 19, no. 7, p. 24, 2017.
[15] G. C. Wu and D. Baleanu, "Chaos synchronization of the discrete fractional logistic map," *Signal Process.*, vol. 102, pp. 96–99, Sep. 2014.
[16] G. C. Wu, D. Baleanu, H. P. Xie, and F. L. Chen, "Chaos synchronization of fractional chaotic maps based on the stability condition," *Physica A Stat. Mech. Appl.*, vol. 460, pp. 374–383, May 2016.
[17] A. A. Khennaoui, A. Ouannas, S. Bendoukha, X. Wang, and V. T. Pham, "On chaos in the fractional-order discrete-time unified system and its control synchronization," *Entropy*, vol. 20, no. 7, p. 3390, 2018.
[18] M. F. Danca, M. Fečkan, and N. Kuznetsov, "Chaos control in the fractional order logistic map via impulses," *Nonlinear Dyn.*, vol. 98, no. 2, pp. 1219–1230, 2019.
[19] V. E. Tarasov and G. M. Zaslavsky, "Fractional equations of kicked systems and discrete maps," *J. Phys. A Math. Theor.*, vol. 41, no. 43, pp. 1–26, 2008.
[20] A. Ouannas, A. A. Khennaoui, S. Momani, G. Grassi, and V. T. Pham, "Chaos and control of a three-dimensional fractional order discrete-time system with no equilibrium and its synchronization," *AIP Adv.*, vol. 10, no. 4, 2020, Art. no. 045310.
[21] A. Ouannas *et al.*, "A quadratic fractional map without equilibria: Bifurcation, 0–1 test, complexity, entropy, and control," *Electronics*, vol. 9, no. 5, p. 748, 2020.
[22] A. Ouannas, A. A. Khennaoui, S. Momani, V. T. Pham, and R. El-Khazali, "Hidden attractors in a new fractional-order discrete system: Chaos, complexity, entropy, and control," *Chin. Phys. B*, vol. 29, no. 5, 2020, Art. no. 050504.
[23] M. F. Danca, "Puu system of fractional order and its chaos suppression," *Symmetry*, vol. 12, no. 3, p. 340, 2020.
[24] L. Franco-Pérez, G. Fernández-Anaya, and L. A. Quezada-Téllez, "On stability of nonlinear nonautonomous discrete fractional Caputo systems," *J. Math. Anal. Appl.*, vol. 487, no. 2, 2020, Art. no. 124021.

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