Lifshitz field theories, Snyder noncomutative space-time and momentum dependent metric

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Abstract

In this work, we propose three different modified relativistic particles. In the first case, we propose a particle with metrics depending on the momenta and we show that the quantum version of these systems includes different field theories, as Lifshitz field theories. As a second case we propose a particle that implies a modified symplectic structure and we show that the quantum version of this system gives different noncommutative space-times, for example the Snyder space-time. In the third case, we combine both structures before mentioned, namely noncommutative space-times and momentum dependent metrics. In this last case, we show that anisotropic field theories can be seen as a limit of noncommutative field theory.

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1 Introduction

Recently, different approaches have been developed to obtain a quantum version of gravity. Some of these approaches are string theory [1], loop quantum gravity [2], noncommutative geometry [3], etc. In $(2+1)$ dimensions there are important progress [4, 5], but in $(3+1)$ dimensions we do not know how this theory is, we only have some signs about it. For instance, using the Ehrenfest principle, Bekenstein proposed that in a quantum gravity the area of the event horizon has discrete spectrum [6, 7]

$$A_n = 4\pi r^2 = \gamma l_p^2 n, \qquad n = 1, 2, \cdots. \tag{1}$$

In addition, G. 't Hooft showed that in $(2+1)$ dimensions the quantum gravity implies a discrete space-time in an effective approximation [8]. For those reasons, we can conjecture that in the quantum gravity there are geometric quantities with discrete spectrum. Remarkably, the discrete space-time obtained by G. 't Hooft is the so-called Snyder space-time, which is discrete, noncommutative and compatible with the Lorentz symmetry [9]. In fact, in this noncommutative space-time the surface area of a sphere is quantized [10]. It is worth mentioning that it is possible construct field theories in some noncommutative space-times [11, 12, 13], but to build a gravity or field theory in a noncommutative space-time, as Snyder space-time, is a very difficult task. Some work about Snyder space-time can be seen in [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27] and works about noncommutative space-times which imply discrete geometric quantities can be seen in [28, 29, 30, 31, 32].

A major problem to obtain a quantum gravity theory is that the usual gravity is unrenormalizable. Nevertheless, recently Hořava formulated a modified gravity which seems to be free ghosts and power counting renormalizable [33]. This gravity is invariant under anisotropic scaling

$$\vec{x} \to b\vec{x}, \qquad t \to b^z t, \quad z, b = \text{constants}, \tag{2}$$

with $z = 3$. The original Hořava gravity has dynamical inconsistencies [34], but were found healthy extensions of it [35, 36]. Hořava gravity has different interesting properties, some of them are [37, 38, 39, 40, 41, 42, 43, 44, 45]. Field theories invariant under the anisotropic scaling transformations (2) can be seen in [46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56], notably these field theories improve their high energy behavior. Furthermore, in the usual general relativity it has been found space-times invariant under the anisotropic scaling (2), see [57]. Significantly, using the gravity/gauge correspondence, these space-times can be related with some condensed matter systems [58, 59, 60].

Now, as a road to obtain new physics, different authors have been proposed that the Minkowski geometry should be changed by the Finsler geometry [61, 62, 63, 64, 65, 66]. In this new geometry the metric depends on velocities, notice that in this case the metric can depends on the momenta. Remarkably, since 1938 Max Born proposed a theory with a metric that depends on the momenta as a suggestion for unifying quantum theory and relativity [67].

In this work, we propose three different modified relativistic particles. In the first case, we propose a particle with a metric depending on the momenta and we show that the quantum version of this system includes different field theories, as anisotropic field theories. As a second case we propose a particle that implies a modified symplectic structure and we show that the quantum version of this system gives different noncommutative space-times, for example the Snyder space-time. In the third case, we combine both structures before mentioned, namely noncommutative space-times and momentum-dependent metric. In this last case, we show that anisotropic field theories can be seen as a limit of noncommutative field theory.

This paper is organized as follow: in Sec. 2, we study the first modified relativistic particle and show that in this framework different Lifshitz field theories can be obtained; in Sec. 3, we propose a modified particle that its quantum version implies noncommutative space-times; in Sec. 4, we combine the results obtained in Sec. 2 and 3. Finally, in Sec. 5, we provide a summary.

2 Modified actions

The Hamiltonian action for the massive relativistic particle with the momenta fixed in the end points is given by

$$S = \int d\tau \left(-\dot{p}_\mu x^\mu - \frac{\lambda}{2}\left(p^2 - m^2\right) \right) = \int d\tau \left(-\eta^{\mu\nu}\dot{p}_\mu x_\nu - \frac{\lambda}{2}\left(\eta^{\mu\nu} p_\mu p_\nu - m^2\right) \right), \quad (3)$$

where λ is a Lagrange multiplier. Now, if we consider a momentum-dependent metric

$$\Omega^{\mu\nu}(p), \tag{4}$$

we can propose the generalized relativistic particle action

$$S = \int d\tau \left(-\eta^{\mu\nu}\dot{p}_\mu x_\nu - \frac{\lambda}{2}\left(\Omega^{\mu\nu}(p)p_\mu p_\nu - m^2\right) \right). \tag{5}$$

In the section 3, we will show that the quantum version of this system includes different field theories, as Lifshitz field theories.

In addition, if

$$\Lambda^{\mu\nu}(p) \tag{6}$$

is a symmetric matrix we can also introduce the alternative action

$$S = \int d\tau \left(-\Lambda^{\mu\nu}(p)\dot{p}_\mu x_\nu - \frac{\lambda}{2}\left(\eta^{\mu\nu}p_\mu p_\nu - m^2\right) \right). \tag{7}$$

Notice that in this case the space-time metric is not modified. In the section 4, we will show that this system has a modified symplectic structure and also we will show that the quantum version of this system gives different noncommutative space-times, for example the Snyder space-time.

Furthermore, we can take the action

$$S = \int d\tau \left(-\Lambda^{\mu\nu}(p)\dot{p}_\mu x_\nu - \frac{\lambda}{2}\left(\Omega^{\mu\nu}(p)p_\mu p_\nu - m^2\right) \right), \tag{8}$$

which contains the actions (5) and (7). In all these cases, we assume that we use the Minkowski metric for raising and lowering indices.
Notice that these three actions are invariant under reparametrization transformations

$$\tau \to \tau(\tilde{\tau}), \qquad \lambda \to \lambda\frac{d\tilde{\tau}}{d\tau},$$

in fact this symmetry appears in the usual relativistic particle [68]. Due that the reparametrization symmetry is a local symmetry, according to Dirac's Method [69, 68] these three action have a first class constraint

$$C(x,p) \approx 0,$$

which generates the reparametrization symmetry (the "gauge symmetry" for these systems). Now, if $A(x,p)$ is a function in the phase space, according to the Dirac's Method, an infinitesimal gauge transformation are given by

$$\delta A = \epsilon(x,p)\{A(x,p), C(x,p)\}.$$

Notice that only if $A(x,p)$ is a gauge invariant quantity we have

$$\{A(x,p), C(x,p)\} = 0.$$

Furthermore, at quantum level Dirac's method sets that the physical states are invariant under the action of the first class constraints, i.e.,

$$\exp(\zeta\hat{C})|\psi\rangle = |\psi\rangle, \tag{9}$$

which implies

$$\hat{C}|\psi\rangle = 0. \tag{10}$$

Here $\hat{C}$ is the quantum version of the constraint $C(x,p)$.

In the next sections we will show that these three actions are different and each one of them gives an alternative quantum physics.

3 Lifshitz case

First we consider the following action

$$S = \int d\tau \left(-\eta^{\mu\nu}\dot{p}_\mu x_\nu - \frac{\lambda}{2}\left(\Omega^{\mu\nu}(p)p_\mu p_\nu - m^2\right)\right), \tag{11}$$

from this action we obtain the classical constraint

$$C = \Omega^{\mu\nu}(p)p_\mu p_\nu - m^2 \approx 0. \tag{12}$$

Then, using the Dirac's Method, at quantum level and in the coordinate representation we get the wave equation

$$\left(-\Omega^{\mu\nu}(-i\partial)\partial_\mu\partial_\nu - m^2\right)\phi = 0, \tag{13}$$

which is a modified Klein-Gordon equation.

3.1 Scalar Field

The equation (13) can be obtained from the action

$$S = \int dx^{d+1} \frac{1}{2} \left(\partial_\mu \phi \Omega^{\mu\nu} \left(-i\partial \right) \partial_\nu \phi - m^2 \phi^2 \right). \tag{14}$$

In fact, if the matrix $\Omega^{\mu\nu} \left(-i\partial \right)$ has only an even number of derivatives, we arrive to

$$\delta S = - \int dx^{d+1} \delta\phi \left(\Omega^{\mu\nu} \left(-i\partial \right) \partial_\mu \partial_\nu \phi + m^2 \phi \right) = 0, \tag{15}$$

which implies the equation of motion

$$\Omega^{\mu\nu} \left(-i\partial \right) \partial_\mu \partial_\nu \phi + m^2 \phi = 0, \tag{16}$$

that is equivalent to the equation (13).

Notice, that if

$$\Omega^{\mu\nu} \left(-i\partial \right) = \eta^{\mu\nu} + h^{\mu\nu} \left(-i\partial \right), \tag{17}$$

we have

$$\begin{aligned} S &= \int dx^{d+1} \frac{1}{2} \left(\partial_\mu \phi \partial^\mu \phi + \left(\partial_\mu \phi \right) h^{\mu\nu} \left(-i\partial \right) \left(\partial_\nu \phi \right) - m^2 \phi^2 \right) \\ &= \int dx^{d+1} \frac{1}{2} \left(\partial_\mu \phi \partial^\mu \phi - \phi h^{\mu\nu} \left(-i\partial \right) \left(\partial_\mu \partial_\nu \phi \right) - m^2 \phi^2 \right). \end{aligned} \tag{18}$$

In particular, if we take

$$\Omega^{\mu\nu}(p) = \eta^{\mu\nu} + l^2 p^\mu p^\nu, \qquad l = \text{constant}, \tag{19}$$

namely

$$\Omega^{\mu\nu} \left(-i\partial \right) = \eta^{\mu\nu} + l^2 \hat{p}^\mu \hat{p}^\nu = \eta^{\mu\nu} - l^2 \partial^\mu \partial^\nu, \tag{20}$$

we arrive to

$$S = \int dx^{d+1} \frac{1}{2} \left(\partial_\mu \phi \partial^\mu \phi + l^2 \phi \Box^2 \phi - m^2 \phi^2 \right). \tag{21}$$

In general, if $G(p^2)$ is a smooth function, when

$$\Omega^{\mu\nu}(p) = \eta^{\mu\nu} + G\left(p^2\right) p^\mu p^\nu, \tag{22}$$

we obtain

$$\Omega^{\mu\nu}\left(-i\partial\right) = \eta^{\mu\nu} - G\left(-\Box\right) \partial^\mu \partial^\nu, \tag{23}$$

for this case, the action (18) becomes

$$S = \int dx^{d+1} \frac{1}{2} \left(\partial_\mu \phi \partial^\mu \phi + \phi G\left(-\Box\right) \Box^2 \phi - m^2 \phi^2\right). \tag{24}$$

This is a quantum field theory with high order time derivatives, which implies the existence of ghost field solutions [70]. To avoid these kind of solutions we introduce the matrix

$$\Omega^{0\mu}(p) = \eta^{0\mu}, \qquad \Omega^{ij}(p) = \delta^{ij} + l^2 p^i p^j, \tag{25}$$

the quantum version of this last equation is given by

$$\Omega^{0\nu}\left(-i\partial\right) = \eta^{0\mu}, \qquad \Omega^{ij}(p) = \delta^{ij} - l^2 \partial^i \partial^j, \tag{26}$$

and for the action (18) we get

$$\begin{aligned} S &= \int dx^{d+1} \frac{1}{2} \left(\partial_\mu \phi \partial^\mu \phi - l^2 \partial_i \phi \partial^i \partial^j \partial_j \phi - m^2 \phi^2\right) \\ &= \int dx^{d+1} \frac{1}{2} \left(\partial_\mu \phi \partial^\mu \phi + l^2 \phi \left(\nabla^2\right)^2 \phi - m^2 \phi^2\right). \end{aligned} \tag{27}$$

This is the action for an anisotropic scalar field with dynamic exponent $z = 2$, also this field is named Lifshitz scalar field [52, 56].

Moreover, for the case

$$\Omega^{0\nu}(p) = \eta^{0\nu}, \qquad \Omega^{ij}(p) = \delta^{ij} + \alpha \left(-\vec{p}^{\,2}\right)^{z-2} p^i p^j, \tag{28}$$

namely

$$\Omega^{0\nu}(-i\partial) = \eta^{0\nu}, \qquad \Omega^{ij}(-i\partial) = \delta^{ij} - \alpha \left(\nabla^2\right)^{z-2} \partial^i \partial^j, \tag{29}$$

the action (18) becomes

$$S = \int dx^{d+1} \frac{1}{2} \left(\partial_\mu \phi \partial^\mu \phi + \alpha \phi \left(\nabla^2 \right)^z \phi - m^2 \phi^2 \right). \quad (30)$$

Which is the action for an anisotropic scalar field with dynamic exponent z, [52, 56]. Then, a Lifshitz scalar field can be seen as a scalar field in a generalized metric depending on the momenta.

In general, if $G(\vec{p}^{\,2})$ is a smooth function, we can propose the matrix

$$\Omega^{0\nu}(p) = \eta^{0\nu}, \qquad \Omega^{ij}(p) = \delta^{ij} + G\left(\vec{p}^{\,2}\right) p^i p^j, \quad (31)$$

which at quantum level is

$$\Omega^{0\nu}\left(-i\partial\right) = \eta^{0\nu}, \qquad \Omega^{ij}\left(-i\partial\right) = \eta^{ij} - G\left(-\nabla^2\right) \partial^i \partial^j, \quad (32)$$

gives the action

$$S = \int dx^{d+1} \frac{1}{2} \left(\partial_\mu \phi \partial^\mu \phi + \phi G\left(-\nabla^2\right) \left(\nabla^2 \right)^2 \phi - m^2 \phi^2 \right). \quad (33)$$

In the next subsections we will study other fields in a momentum dependent metric.

3.2 Dirac Field

Now we construct a Dirac equation in a metric that depends on the momenta. In this case, we require a tetrad formalism associate to the momenta dependent metric. For this reason, let us introduce the tetrad

$$e^\mu_a = e^\mu_a\left(-i\partial\right), \quad (34)$$

which satisfies

$$e^\mu_a\left(-i\partial\right) e^\nu_b\left(-i\partial\right) \eta^{ab} = \Omega^{\mu\nu}\left(-i\partial\right). \quad (35)$$

Then, using the usual Dirac's matrices, that satisfy the ordinary Clifford algebra

$$\left\{ \gamma^a, \gamma^b \right\}_+ = 2\eta^{ab}, \quad (36)$$

and the tetrad basis introduced in (34) we obtain the following matrices

$$\Gamma^{\mu}\left(-i\partial\right) = e_{a}^{\mu}\left(-i\partial\right)\gamma^{a}, \tag{37}$$

which satisfy

$$\left\{\Gamma^{\mu}\left(-i\partial\right), \Gamma^{\nu}\left(-i\partial\right)\right\}_{+} = 2\Omega^{\mu\nu}\left(-i\partial\right). \tag{38}$$

With the matrices Γ^{μ} we propose the modified Dirac equation

$$-i\Gamma^{\mu}\partial_{\mu}\psi + m\psi = 0. \tag{39}$$

Notice that using this last equation and (38), we arrive to

$$\begin{aligned}
\left(-i\Gamma^{\nu}\partial_{\nu} - m\right)\left(-i\Gamma^{\mu}\partial_{\mu} + m\right)\psi &= \\
&= -\Gamma^{\mu}\Gamma^{\nu}\partial_{\mu}\partial_{\nu}\psi - m^{2}\psi \\
&= \left(-\frac{\Gamma^{\mu}\Gamma^{\nu} + \Gamma^{\nu}\Gamma^{\mu}}{2}\partial_{\mu}\partial_{\nu} - m^{2}\right)\psi = 0,
\end{aligned}$$

namely

$$-\left(\Omega^{\mu\nu}\left(-i\partial\right)\partial_{\mu}\partial_{\nu} + m^{2}\right)\psi = 0, \tag{40}$$

which is the modified Klein-Gordon equation (13). Then, the generalized Dirac's equation can be seen as a Dirac's equation in a metric depending on the momenta.

In particular, if we take

$$\Omega^{\mu\nu}\left(-i\partial\right) = \eta^{\mu\nu} + h^{\mu\nu}\left(-i\partial\right), \tag{41}$$

at first order, the tetrad results

$$e_{a}^{\mu}\left(-i\partial\right) = \eta_{a}^{\mu} + \frac{1}{2}h_{a}^{\mu}\left(-i\partial\right), \tag{42}$$

which satisfies

$$e_{a}^{\mu}e_{b}^{\nu}\eta^{ab} \approx \eta^{\mu\nu} + h^{\mu\nu}\left(-i\partial\right). \tag{43}$$

In this approximation we obtain

$$\Gamma^{\mu} = \gamma^{\mu} + \frac{1}{2}h_{a}^{\mu}\gamma^{a} \tag{44}$$

and the modified Dirac's equation is given by

$$\left(-i\gamma^{\mu}\partial_{\mu}+m\right)\psi-i\frac{1}{2}h_{a}^{\mu}\gamma^{a}\partial_{\mu}\psi=0. \tag{45}$$

For the case

$$h^{\mu\nu}\left(-i\partial\right)=-G\left(-\Box\right)\partial^{\mu}\partial^{\nu}, \tag{46}$$

we arrive to

$$\left(-i\gamma^{\mu}\partial_{\mu}+m\right)\psi+\frac{i}{2}G\left(-\Box\right)\partial^{\mu}\partial_{a}\gamma^{a}\partial_{\mu}\psi=0, \tag{47}$$

namely

$$\left(-i\gamma^{\mu}\partial_{\mu}+m\right)\psi+\frac{i}{2}G\left(-\Box\right)\Box\gamma^{\mu}\partial_{\mu}\psi=0. \tag{48}$$

While, if we take

$$h^{\mu 0}\left(-i\partial\right)=0, \qquad h^{ij}\left(-i\partial\right)=-G\left(-\nabla^{2}\right)\partial^{i}\partial^{j}, \tag{49}$$

we get

$$\left(-i\gamma^{\mu}\partial_{\mu}+m\right)\psi+\frac{i}{2}G\left(-\nabla^{2}\right)\nabla^{2}\gamma^{\mu}\partial_{\mu}\psi=0. \tag{50}$$

In particular when

$$G\left(-\nabla^{2}\right)=\alpha\left(\nabla^{2}\right)^{z-2}, \qquad \alpha=\text{constant}, \tag{51}$$

we obtain

$$\left(-i\gamma^{\mu}\partial_{\mu}+m\right)\psi+\frac{i}{2}\alpha\left(-\nabla^{2}\right)^{z-1}\gamma^{\mu}\partial_{\mu}\psi=0, \tag{52}$$

which is the anisotropic Dirac's equation with dynamic exponent z, [46, 56]. Therefore, the anisotropic Dirac's equation can be seen as a Dirac's equation in a metric depending on the momenta.

3.3 Electromagnetic Field

For the electromagnetic field we propose the action

$$S=-\frac{1}{4}\int dx^{d+1}F_{\mu\nu}\Omega^{\mu\alpha}\left(-i\partial\right)\Omega^{\nu\beta}\left(-i\partial\right)F_{\alpha\beta}. \tag{53}$$

Notice that if

$$\Omega^{\mu\nu}\left(-i\partial\right)=\eta^{\mu\nu}+h^{\mu\nu}\left(-i\partial\right), \tag{54}$$

at first order, we obtain

$$\begin{aligned}\Omega^{\mu\alpha}\left(-i\partial\right)\Omega^{\nu\beta}\left(-i\partial\right) &= \left(\eta^{\mu\alpha}+h^{\mu\alpha}\right)\left(\eta^{\nu\beta}+h^{\nu\beta}\right)\\ &\approx \eta^{\mu\alpha}\eta^{\nu\beta}+h^{\mu\alpha}\left(-i\partial\right)\eta^{\mu\beta}+\eta^{\mu\alpha}h^{\nu\beta}\left(-i\partial\right),\end{aligned} \tag{55}$$

which implies

$$F_{\mu\nu}\Omega^{\mu\alpha}\left(-i\partial\right)\Omega^{\nu\beta}\left(-i\partial\right)F_{\alpha\beta}\approx F_{\mu\nu}F^{\mu\nu}+2F_{\mu}{}^{\beta}h^{\mu\alpha}\left(-i\partial\right)F_{\alpha\beta}. \tag{56}$$

Moreover, using this last equation we arrive to

$$S=-\frac{1}{4}\int dx^{d+1}\left(F_{\mu\nu}F^{\mu\nu}+2F_{\mu}{}^{\beta}h^{\mu\alpha}\left(-i\partial\right)F_{\alpha\beta}\right). \tag{57}$$

In particular, when

$$h^{\mu\nu}\left(-i\partial\right)=-G\left(-\Box\right)\partial^{\mu}\partial^{\nu}, \tag{58}$$

we get the action

$$\begin{aligned}S &= -\frac{1}{4}\int dx^{d+1}\left(F_{\mu\nu}F^{\mu\nu}-2F_{\mu}{}^{\beta}G\left(-\Box\right)\partial^{\mu}\partial^{\alpha}F_{\alpha\beta}\right)\\ &= \int dx^{d+1}\left(F_{\mu\nu}F^{\mu\nu}+2\partial^{\mu}F_{\mu}{}^{\beta}G\left(-\Box\right)\partial^{\alpha}F_{\alpha\beta}\right).\end{aligned} \tag{59}$$

Furthermore, if $h^{\mu\nu}$ is given by

$$h^{\mu0}\left(-i\partial\right)=0,\qquad h^{ij}\left(-i\partial\right)=-G\left(-\nabla^{2}\right)\partial^{i}\partial^{j}, \tag{60}$$

the following action

$$S = -\frac{1}{4}\int dx^{d+1}\left(F_{\mu\nu}F^{\mu\nu}+2\partial_{i}F_{ik}G\left(-\nabla^{2}\right)\partial_{j}F_{jk}\right) \tag{61}$$

is obtained.

Notice that, when $G\left(-\nabla^2\right) = l^2$, we arrive to the action

$$S = -\frac{1}{4}\int dx^{d+1}\left(F_{\mu\nu}F^{\mu\nu} + 2l^2\partial_i F_{ik}\partial_j F_{jk}\right), \tag{62}$$

which is the action for the anisotropic electrodynamics field with dynamic exponent $z = 2$, [48]. Then, this last system can be seen as an electrodynamics in a metric depending on the momenta.

3.4 Yang-Mills Field

It is well known that for non abelian fields the usual derivative is changed by the covariant derivative, in the sense that the partial derivative is not gauge covariant. For this reason, for non abelian gauge fields the matrix $\Omega^{\mu\nu}$ is changed by

$$\Omega^{\mu\nu}(-i\partial) \to \Omega^{\mu\alpha}\left(-iD\right), \tag{63}$$

where

$$D_\mu\mathcal{F} = \partial_\mu\mathcal{F} + [\mathcal{A}_\mu, \mathcal{F}], \tag{64}$$

for $\mathcal{F}$ in a matrix representation of the Lie group. Hence, in this framework the usual Yang-Mills action is changed by

$$S = \frac{1}{2g^2}\int dx^{d+1}Tr\left(\mathcal{F}_{\mu\nu}\Omega^{\mu\alpha}\left(-iD\right)\Omega^{\nu\beta}\left(-iD\right)\mathcal{F}_{\alpha\beta}\right). \tag{65}$$

which is gauge invariant. In particular when

$$\Omega^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}\left(-iD\right), \tag{66}$$

at first order, we get

$$S = \frac{1}{2g^2}\int dx^{d+1}Tr\left(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + 2\mathcal{F}_\mu{}^\beta h^{\mu\alpha}\left(-iD\right)\mathcal{F}_{\alpha\beta}\right). \tag{67}$$

Moreover, if

$$h^{\mu 0}\left(-iD\right) = 0, \qquad h^{ij}\left(-iD\right) = -l^2 D^i D^j, \tag{68}$$

we arrive to

$$
\begin{aligned}
S &= \frac{1}{2}g^2 \int dx^{d+1} Tr\left(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - l^2 \mathcal{F}_i{}^k D^i D^j \mathcal{F}_{jk}\right) \\
&= \frac{1}{2}g^2 \int dx^{d+1} Tr\left(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + l^2 D_i \mathcal{F}_{ik} D_j \mathcal{F}_{jk}\right), \qquad (69)
\end{aligned}
$$

which is the action for the anisotropic Yang-Mills with dynamic exponent $z = 2$, [52]. Therefore, this anisotropic Yang-Mills theory can be seen as a Yang-Mills theory in a momentum dependent metric.

4 Noncommutative case

In this section we will consider the Hamiltonian action

$$
S = \int d\tau \left(-\Lambda^{\mu\nu}(p)\dot{p}_\mu x_\nu - H(x,p)\right). \qquad (70)
$$

This action implies the equations of motion

$$
\begin{aligned}
\dot{x}^\mu &= \left(\Lambda^{-1}\right)^{\mu\alpha}\left(\frac{\partial \Lambda_{\alpha\rho}}{\partial p^\nu} - \frac{\partial \Lambda_{\nu\rho}}{\partial p^\alpha}\right) x^\rho \left(\Lambda^{-1}\right)^{\nu\gamma}\frac{\partial H}{\partial x^\gamma} + \left(\Lambda^{-1}\right)^{\mu\gamma}\frac{\partial H}{\partial p^\gamma}, \qquad (71) \\
\dot{p}^\mu &= -\left(\Lambda^{-1}\right)^{\mu\alpha}\frac{\partial H}{\partial x^\alpha}. \qquad (72)
\end{aligned}
$$

It can be shown that these equations of motion are consistent with the symplectic structure

$$
\begin{aligned}
\{x^\mu, x^\nu\} &= \left(\Lambda^{-1}\right)^{\mu\alpha}\left(\frac{\partial \Lambda_{\alpha\rho}}{\partial p^\beta} - \frac{\partial \Lambda_{\beta\rho}}{\partial p^\alpha}\right) x^\rho \left(\Lambda^{-1}\right)^{\beta\nu}, \qquad (73) \\
\{x^\mu, p^\nu\} &= \left(\Lambda^{-1}\right)^{\mu\nu}, \qquad (74) \\
\{p^\mu, p^\nu\} &= 0. \qquad (75)
\end{aligned}
$$

Notice that the quantum version of this symplectic structure is a noncommutative space-time.

Now, we can see that if H does not depend on x, the equations of motion (71)-(72) are

$$
\begin{aligned}
\dot{x}_\mu &= \left(\Lambda^{-1}\right)^{\mu\gamma}\frac{\partial H}{\partial p^\gamma}, \qquad (76) \\
\dot{p}_\mu &= 0, \qquad (77)
\end{aligned}
$$

namely,

$$\ddot{x}_\mu = 0. \tag{78}$$

Then, in all noncommutative space-time, the classical free particle follows the standard dynamics.

We can see that if the coordinate $(\tilde{x}^\mu, \tilde{p}_\mu)$ satisfy the usual Poisson brackets, then the phase space coordinates

$$x^\mu = \left(\Lambda^{-1}\right)^{\mu\nu} \tilde{x}_\nu, \qquad \tilde{p}_\mu = p_\mu, \tag{79}$$

satisfy the relations (73)-(75), this is the so-called Darboux mapping, which is only locally defined.

If we take the modified action for the free relativistic particle

$$S = \int d\tau \left(-\Lambda^{\mu\nu}(p)\dot{p}_\mu x_\nu - \frac{\lambda}{2}\left(\eta^{\mu\nu}p_\mu p_\nu - m^2\right)\right), \tag{80}$$

we obtain the usual constraint

$$p^2 - m^2 \approx 0. \tag{81}$$

Notice that we are working in the extended phase space [69] where all the momenta are independent and are not restricted by the constraint (81). In this way, the symplectic structure will be invertible.

When in a noncommutative space-time there are no constant parameters in its commutation relations, construct an interacting field theory in that space-time is a very difficult task. Some advances in this topic can be seen in [71] and some proposals to obtain a field theory in the noncommutative Snyder space-time can be seen in [17, 20]. Construct a field theory in a noncommutative space-time is not the main issue of this paper. However, in different noncommutative space-times, the free particle is a special case. In fact, in some noncommutative space-times the free scalar field is not different form the free scalar field in a commutative space-time [13, 72]. Notice that using local Darboux coordinates (79), the equation

$$\left(-\tilde{\partial}^\mu \tilde{\partial}_\mu - m^2\right)\phi = 0 \tag{82}$$

can be proposed as a quantum version of the constraints (81). In the next section we will propose different cases of $\Lambda^{\mu\nu}(p)$.

4.1 Snyder space-time

Snyder space-time is an important example of a noncommutative space-time. For this case, if we take the following matrix

$$\left(\Lambda^{-1}\right)^{\mu\nu}(p) = \eta^{\mu\nu} + a^2 p^\mu p^\nu, \qquad a = \text{constant}, \tag{83}$$

we obtain the symplectic structure

$$\begin{aligned} \{x^\mu, x^\nu\} &= a^2 \left(x^\mu p^\nu - x^\nu p^\mu\right), &(84)\\ \{x^\mu, p^\nu\} &= \eta^{\mu\nu} + a^2 p^\mu p^\nu, &(85)\\ \{p^\mu, p^\nu\} &= 0. &(86)\end{aligned}$$

This symplectic structure is compatible with the noncommutative Snyder space-time [9]:

$$\begin{aligned} [\hat{x}^\mu, \hat{x}^\nu] &= ia^2 \left(\hat{x}^\mu \hat{p}^\nu - \hat{x}^\nu \hat{p}^\mu\right), &(87)\\ [\hat{x}^\mu, \hat{p}^\mu] &= i\left(\eta^{\mu\nu} + a^2 \hat{p}^\mu \hat{p}^\nu\right), &(88)\\ [\hat{p}^\mu, \hat{p}^\mu] &= 0. &(89)\end{aligned}$$

In general, if $f(p^2)$ is a smooth function, we can propose

$$\left(\Lambda^{-1}\right)^{\mu\nu}(p) = \eta^{\mu\nu} + f(p^2) p^\mu p^\nu, \tag{90}$$

which implies the symplectic structure

$$\begin{aligned} \{x^\mu, x^\nu\} &= f(p^2) \left(x^\mu p^\nu - x^\nu p^\mu\right), &(91)\\ \{x^\mu, p^\nu\} &= \eta^{\mu\nu} + f(p^2) p^\mu p^\nu, &(92)\\ \{p^\mu, p^\nu\} &= 0. &(93)\end{aligned}$$

The quantum version of these last Poisson brackets imply a noncommutative space-time.

4.2 Euclidean Snyder space-time

Furthermore, if we take

$$\left(\Lambda^{-1}\right)^{00} = -1, \quad \left(\Lambda^{-1}\right)^{0i} = 0, \quad \left(\Lambda^{-1}\right)^{ij} = \delta^{ij} + a^2 p^i p^j, \tag{94}$$

we have the following Poisson brackets:

$$\begin{aligned} \left\{x^0, x^\nu\right\} &= 0, & (95)\\ \left\{x^i, x^j\right\} &= a^2\left(x^i p^j - x^j p^i\right), & (96)\\ \left\{x^0, p^\mu\right\} &= \eta^{0\mu}, & (97)\\ \left\{x^i, p^j\right\} &= \delta^{ij} + a^2 p^i p^j, & (98)\\ \left\{p^\mu, p^\nu\right\} &= 0. & (99)\end{aligned}$$

It can be show that the quantum version of this space-time implies discrete geometric quantities [10].

In general, if $f(\vec{p}^{\,2})$ is a smooth function, we can propose the matrix

$$\left(\Lambda^{-1}\right)^{00} = -1, \quad \left(\Lambda^{-1}\right)^{0i} = 0, \quad \left(\Lambda^{-1}\right)^{ij} = \delta^{ij} + f(\vec{p}^{\,2}) p^i p^j, \qquad (100)$$

which implies the Poisson brackets

$$\begin{aligned} \left\{x^0, x^\nu\right\} &= 0, & (101)\\ \left\{x^i, x^j\right\} &= f(\vec{p}^{\,2})\left(x^i p^j - x^j p^i\right), & (102)\\ \left\{x^0, p^\mu\right\} &= \eta^{0\mu}, & (103)\\ \left\{x^i, p^j\right\} &= \delta^{ij} + f(\vec{p}^{\,2}) p^i p^j, & (104)\\ \left\{p^\mu, p^\mu\right\} &= 0. & (105)\end{aligned}$$

The quantum version of these Poisson brackets implies a noncommutative space-time.

5 Noncommutative space-time and Lifshitz field theory

The modified free particle given by

$$S = \int d\tau \left(-\Lambda^{\mu\nu}(p)\dot{p}_\mu x_\nu - \frac{\lambda}{2}\left(\Omega^{\mu\nu}(p) p_\mu p_\nu - m^2\right)\right) \qquad (106)$$

includes both modified free particles studied before. For this system we have the constraint

$$\Omega^{\mu\nu}(p) p_\mu p_\nu - m^2 \approx 0. \qquad (107)$$

and the symplectic structure (73)-(75). Notice that the classical equations of motion for this system are the classical equations of motion for the usual classical relativistic free particle. However, in suitable local Darboux coordinates (79), we get the wave equation

$$\left(-\Omega^{\mu\nu}\left(-i\tilde{\partial}\right)\tilde{\partial}_{\mu}\tilde{\partial}_{\nu}-m^{2}\right)\phi=0, \tag{108}$$

as the quantum version of the constraint (107). This last equation is a modified Klein-Gordon equation. Notice that the quantum propagation of this free particle is changed.

We do not know how an interacting field theory in this new framework is. However, in the limit

$$\Lambda^{\mu\nu}(p)\rightarrow\eta^{\mu\nu}, \tag{109}$$

for non trivial $\Omega^{\mu\nu}(p)$, we should obtain a Lifshitz-like field theory. In addition, in the limit

$$\Omega^{\mu\nu}(p)\rightarrow\eta^{\mu\nu} \tag{110}$$

we should obtain an usual interacting field theory in a noncommutative space-time.

In the usual particle, the Minkowski metric appears in the term $\eta^{\mu\nu}\dot{p}_{\mu}x_{\nu}$ and in the term $\eta^{\mu\nu}p_{\mu}p_{\nu}$. Then in this case we have

$$\Omega^{\mu\nu}(p)=\eta^{\mu\nu}=\Lambda^{\mu\nu}(p). \tag{111}$$

For this reason, we can take the modified action

$$S=\int d\tau\left(-\Omega^{\mu\nu}(p)\dot{p}_{\mu}x_{\nu}-\frac{\lambda}{2}\left(\Omega^{\mu\nu}(p)p_{\mu}p_{\nu}-m^{2}\right)\right) \tag{112}$$

as a special case. This last system is interesting, because a Lifshitz field theory is related with a noncommutative space-time. In the next subsection we study some cases.

5.1 Snyder space-time

If we take the matrix

$$\left(\Omega^{-1}\right)^{\mu\nu}(p) = \left(\Lambda^{-1}\right)^{\mu\nu}(p) = \eta^{\mu\nu} + l^2 p^\mu p^\nu, \tag{113}$$

we obtain the Snyder noncommutative space-time (84)-(86) and the constraint (107) is given by

$$p^2 - m^2 - \frac{l^2}{1 + l^2 p^2}(p^2)^2 = 0. \tag{114}$$

At first order, this constraint can be written as

$$p^2 - m^2 - l^2 (p^2)^2 = 0. \tag{115}$$

Then, at this order, the wave equation for this system is given by

$$\left(-\tilde{\partial}_\mu \tilde{\partial}^\mu - m^2 - l^2 (\tilde{\partial}^\mu \tilde{\partial}_\mu)^2\right)\phi = 0. \tag{116}$$

In general, if $f(p^2)$ is a smooth function, we can propose

$$\left(\Omega^{-1}\right)^{\mu\nu}(p) = \left(\Lambda^{-1}\right)^{\mu\nu}(p) = \eta^{\mu\nu} + f(p^2) p^\mu p^\nu, \tag{117}$$

which implies the symplectic structure (91)-(93). For this case, the constraint (107) is given by

$$p^2 - m^2 - \frac{f(p^2)}{1 + f(p^2) p^2}(p^2)^2 = 0. \tag{118}$$

At first order, this constraint can be written as

$$p^2 - m^2 - f(p^2)(p^2)^2 = 0. \tag{119}$$

At quantum level, this constraint implies the wave equation

$$\left(-\tilde{\partial}_\mu \tilde{\partial}^\mu - m^2 - f\left(-\tilde{\Box}\right)(\tilde{\partial}^\mu \tilde{\partial}_\mu)^2\right)\phi = 0. \tag{120}$$

5.2 Euclidean Snyder space-time and Lifshitz field theory

Now, if we take

$$\left(\Omega^{-1}\right)^{00} = \left(\Lambda^{-1}\right)^{00} = -1, \quad \left(\Omega^{-1}\right)^{0i} = \left(\Lambda^{-1}\right)^{0i} = 0,$$
$$\left(\Omega^{-1}\right)^{ij} = \left(\Lambda^{-1}\right)^{ij} = \delta^{ij} + f(\vec{p}^{\,2})p^i p^j,$$

we have the Poisson brackets (101)-(105) and the constraint

$$p^2 - m^2 - \frac{f(\vec{p}^{\,2})}{1 + f(\vec{p}^{\,2})\vec{p}^{\,2}}(\vec{p}^{\,2})^2 = 0. \tag{121}$$

At first order, this constraint can be written as

$$p^2 - m^2 - f(\vec{p}^{\,2})(\vec{p}^{\,2})^2 = 0. \tag{122}$$

Then, at this order, the wave equation for this system is given by

$$\left(-\tilde{\partial}_\mu \tilde{\partial}^\mu - m^2 - f\left(-\tilde{\nabla}^2\right)(\tilde{\nabla}^2)^2\right)\phi = 0. \tag{123}$$

In particular, if

$$f(\vec{p}^{\,2}) = a(-)^{z-1}(\vec{p}^{\,2})^{z-2}, \tag{124}$$

the wave equation (123) becomes

$$\left(-\tilde{\partial}_\mu \tilde{\partial}^\mu - m^2 - a(\tilde{\nabla}^2)^z\right)\phi = 0, \tag{125}$$

which is the equation of motion for a scalar Lifshitz field theory [56]. In addition, this system lives in the following noncommutative space-time:

$$\left[\hat{x}^0, \hat{x}^\nu\right] = 0, \tag{126}$$
$$\left[\hat{x}^i, \hat{x}^j\right] = ia(-)^{z-1}\left(\hat{p}^2\right)^{z-2}\left(\hat{x}^i\hat{p}^j - \hat{x}^j\hat{p}^i\right), \tag{127}$$
$$\left[\hat{x}^0, \hat{p}^\mu\right] = \eta^{0\mu}, \tag{128}$$
$$\left[\hat{x}^i, \hat{p}^j\right] = i\delta^{ij} + ia(-)^{z-1}\left(\hat{p}^2\right)^{z-2}\hat{p}^i\hat{p}^j, \tag{129}$$
$$\left[\hat{p}^\mu, \hat{p}^\mu\right] = 0. \tag{130}$$

Notice that when $z = 2$, this noncommutative space-time implies discrete geometric quantities [10].

6 Summary

In this work, we proposed three different modified first order actions for relativistic particles. In the first case we proposed a particle with a momentum dependent metric and we showed that the quantum version of these systems include different field theories, as Lifshitz field theories. As a second case we proposed a particle that implies a modified symplectic structure and we showed that the quantum version of this system gives different noncommutative space-times, for example the Snyder space-time. In the third case, we combined both structures before mentioned, namely noncommutative space-times and momentum dependent metric. In this last case, we showed that anisotropic field theories can be seen as a limit of noncommutative field theory.

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