

OPEN ACCESS

A relation between massive scalar field in AdS_{d+1} and diffusion in channels

To cite this article: Juan M Romero and Alejandro Gaona 2014 *J. Phys.: Conf. Ser.* **512** 012028

View the [article online](#) for updates and enhancements.

Related content

- [Entropy production and rectification efficiency in colloid transport along a pulsating channel](#)
M Florencia Carusela and J Miguel Rubi
- [Current and efficiency optimization under oscillating forces in entropic barriers](#)
Ferhat Nutku and Ekrem Aydner
- [Force-driven transport through periodic entropy barriers](#)
N. Laachi, M. Kenward, E. Yariv et al.



IOP | ebooks™

Bringing together innovative digital publishing with leading authors from the global scientific community.

Start exploring the collection—download the first chapter of every title for free.

A relation between massive scalar field in AdS_{d+1} and diffusion in channels

Juan M. Romero, Alejandro Gaona

Departamento de Matemáticas Aplicadas y Sistemas, Universidad Autónoma
Metropolitana-Cuajimalpa, México, D.F 01120, México

E-mail: jromero@correo.cua.uam.mx

Abstract. It is shown that, when the diffusion coefficient is a constant and is taken a particular family of channels, the Fick-Jacobs equation is invariant under conformal symmetry. In addition, using the diffusion coefficient and the geometric parameters of the channels, a representation for the conformal algebra is obtained. Furthermore, it is found that for these systems the Fick-Jacobs equation is equivalent to the Schrödinger equation for the 1-dimensional conformal quantum mechanics. Moreover, using this equivalence, it is found a relation between a massive scalar field equation in AdS_{d+1} background and Fick-Jacobs equation, where the geometric parameter of the channels and the geometric parameters of AdS_{d+1} are identified.

1. Introduction

Anti-de Sitter space AdS_{d+1} is the maximally symmetric solution of Einstein's equations with a negative cosmological constant. This space can be represented as the hyperboloid

$$X_0^2 + X_{d+1}^2 - \sum_{i=1}^{d-1} X_i^2 = R^2 \quad (1)$$

in the flat $d + 2$ -dimensional space with metric

$$ds^2 = -dX_0^2 - dX_{d+1}^2 + \sum_{i=1}^{d-1} X_i^2. \quad (2)$$

In Poincaré coordinates, the AdS_{d+1} metric takes the form [1, 2]

$$ds^2 = \frac{R^2}{z^2} (\eta_{\mu\nu} dx^\mu dx^\nu + dz^2), \quad \mu = 0, 1, \dots, d \quad (3)$$

where $\eta_{\mu\nu}$ is the Minkowski metric and z is the so call holographic coordinate. In addition the cosmological constant is given by

$$\Lambda = -\frac{d(d-1)}{2R^2}. \quad (4)$$

In this space, the equation of motion for a massive scalar field, $\phi(x, z)$, is

$$\frac{1}{R^2} \frac{\partial^2 \phi(x, z)}{\partial z^2} + \frac{1}{R^2} \partial^\mu \partial_\mu \phi(x, z) - \frac{(d-1)}{R^2 z} \frac{\partial \phi(x, z)}{\partial z} - \frac{m^2}{z^2} \phi(x, z) = 0. \quad (5)$$



An amazing result in physics is given by the AdS_{d+1}/CFT_d correspondence, which allows a relation between $(d + 1)$ -dimensional gravitational theory and certain classes of d -dimensional Yang-Mills theories [1, 2]. Recently this correspondence has been extended, for example there is a correspondence between a gravitational theory and condensed matter AdS/CMT [5] and also there is a duality between a gravitational theory and fluid dynamics *gravity/fluid* [6, 7]. Furthermore, Brownian motion has been studied in this frame-work [8, 9]. In the AdS_{d+1}/CFT_d correspondence, in order to get a conformal field theory the limit $z \rightarrow 0$ is taken [1, 2], then the holographic coordinate disappears in the CFT_d . However, the holographic coordinate is important too, for example if we take

$$\phi(x, z) = e^{-ip \cdot x} z^{\frac{(d-1)}{2}} \psi(z), \quad p^2 = -M_d^2, \quad (6)$$

the equation (5) becomes

$$\left(-\frac{1}{R^2} \frac{\partial^2}{\partial z^2} + \frac{g}{z^2} \right) \psi(z) = \frac{M_d^2}{R^2} \psi(z), \quad g = \frac{1}{R^2} \left(\left(\frac{d-1}{2} \right) \left(\frac{d+1}{2} \right) + m^2 R^2 \right). \quad (7)$$

Using this last equation, it is possible to get the spectrum of the light hadrons, where z is related with the invariant separation of the quark constituents [3, 4]. Now, the equation (7) is the static Schrödinger equation for the conformal quantum mechanics [34], which appears in different contexts, from black-holes to atomic physics [35, 36, 37]. Notice that, if a system is related with the conformal quantum mechanics, then it is also related with a massive scalar field in AdS_{d+1} .

The conformal group is the symmetry group for both Anti-de Sitter space and conformal field theory, for this reason the conformal symmetry is important in the AdS_{d+1}/CFT_d correspondence [2]. This symmetry appears in other systems, for example the free Schrödinger equation is invariant under a non-relativistic conformal transformations, which is known as the Schrödinger group [10, 11]. In fact this group has been important to study the non-relativistic AdS_{d+1}/CFT_d correspondence [12, 13]. Some work about the Schrödinger group can be seen in [14, 15, 16, 17, 18, 19, 20, 21]. Now, the simplest model of diffusion is described by the Fick equation and Sophus Lie showed that this equation is invariant under the Schrödinger group [22]. Another study about diffusion phenomena and the Schrödinger group can be seen in [23].

When the diffusion is in a channel, which has the shape of surface of revolution with cross sectional area $A(x)$, the Fick equation has to be changed to the Fick-Jacobs equation [24]

$$\frac{\partial C(x, t)}{\partial t} = \frac{\partial}{\partial x} \left[D(x) A(x) \frac{\partial}{\partial x} \left(\frac{C(x, t)}{A(x)} \right) \right], \quad (8)$$

where $C(x, t)$ is the particle concentration and $D(x)$ is the diffusion coefficient. This last equation is important to study diffusion in biological channels, zeolites and nano-channels [25, 26, 27, 28, 29, 30, 31]. In this paper we will show that, when the diffusion coefficient is a constant and the cross sectional area is $A(x) = (b + \lambda x)^{2\nu}$ (where ν, b , and λ are real constants), the Fick-Jacobs equation is equivalent to the conformal quantum mechanics. In addition, a relation between a massive scalar field equation in AdS_{d+1} background and Fick-Jacobs equation is found, in which the diffusion coefficient and cosmological constant for AdS_{d+1} space are associated. Furthermore, it is found that the geometric parameter of the channels and the geometric parameters of AdS_{d+1} space are related too. In this case the axial coordinate of the channel x and the holographic coordinate z of AdS_{d+1} are identified.

This paper is organized in the following way: in section 2 a brief review about the conformal quantum mechanics is given; in section 3 it is shown that the Fick-Jacobs equation is invariant under conformal symmetry for a family of channels; in section 4 it is shown that the Fick-Jacobs equation is equivalent to the conformal quantum mechanics for a set of particular channels and an exact solution for this equation is given. In section 5 a relation between Fick-Jacobs equation and a massive scalar field in AdS_{d+1} is found. Finally, in section 6 a summary is given.

2. Conformal Quantum Mechanics

The Schrödinger equation for the 1-dimensional conformal quantum mechanics is given by

$$i\hbar \frac{\partial \psi(x, t)}{\partial t} = H\psi(x, t) \quad H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{g}{x^2}, \quad (9)$$

which is invariant under the non-relativistic conformal symmetry

$$t' = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad x' = \frac{ax}{\gamma t + \delta}, \quad a^2 = \alpha\delta - \beta\gamma \neq 0, \quad (10)$$

in order to keep the wave equation invariant under the last transformation the wave function must transform like

$$\psi'(x', t') = \sqrt{\gamma t + \delta} e^{i\frac{m}{2\hbar}\Phi(x, t)} \psi(x, t) \quad (11)$$

where

$$\Phi(x, t) = -\frac{\gamma x^2}{\gamma t + \delta}. \quad (12)$$

In this case the group generators are given by H and

$$K_1 = tH - \frac{1}{2} \left(xp - \frac{i}{2} \right), \quad (13)$$

$$K_2 = t^2 H - t \left(xp - \frac{i}{2} \right) + \frac{mx^2}{2} \quad (14)$$

and the algebra

$$[H, K_1] = iH, \quad [H, K_2] = 2iK_1, \quad [K_1, K_2] = iK_2 \quad (15)$$

is satisfied. Using this algebra, it is possible to show that the operator H, K_1, K_2 are conserved.

The classical system with the general potential $V(r) = gr^z$ was first studied by Jacobi [32] and the quantum system with $z = -2$ was originally proposed by Jackiw [33]. The spectrum of the Hamiltonian (9) was found by de Alfaro, Fubini and Furlan [34]. It is worth to mention that the conformal quantum mechanics appears in different contexts, from black-holes to atomic physics [35, 36, 37] and has been proposed as the CFT_1 dual to AdS_2 [38, 39]. We will show that this Hamiltonian appears in diffusion phenomena too.

3. Conformal symmetry and Fick-Jacobs equation

Now, we will study the family of channels with cross sectional area

$$A(x) = (b + \lambda x)^{2\nu}, \quad (\nu, b, \lambda = \text{constant}). \quad (16)$$

In this case, taking $D(x) = D_0$ as a constant and using the change of variable

$$y = b + \lambda x, \quad (17)$$

the Fick-Jacobs equation (8) becomes

$$\frac{\partial C(y, t)}{\partial t} = \lambda^2 D_0 \left(\frac{\partial^2}{\partial y^2} - \frac{2\nu}{y} \frac{\partial}{\partial y} + \frac{2\nu}{y^2} \right) C(y, t). \quad (18)$$

This equation is invariant under conformal transformation (10), where the concentration transforms as

$$C'(y', t') = (\gamma t + \delta)^{\frac{1-2\nu}{2}} e^{-\frac{1}{4D_0} \Phi(y, t)} C(y, t). \quad (19)$$

and Φ is given by (12). If we take $p = -i \frac{\partial}{\partial y}$ the equation (18) can be written like

$$\frac{\partial C(y, t)}{\partial t} = \mathbf{H} C(y, t), \quad (20)$$

where

$$\mathbf{H} = \lambda^2 D_0 \left(-p^2 - i \frac{2\nu}{y} p + \frac{2\nu}{y^2} \right). \quad (21)$$

Now, using

$$\mathbf{K}_1 = t\mathbf{H} - \frac{1}{2} \left(yp - \frac{i}{2} + i\nu \right), \quad (22)$$

$$\mathbf{K}_2 = t^2\mathbf{H} - t \left(yp - \frac{i}{2} + i\nu \right) - \frac{y^2}{4\lambda^2 D_0}. \quad (23)$$

we obtained the conformal algebra

$$[\mathbf{H}, \mathbf{K}_1] = i\mathbf{H}, \quad [\mathbf{H}, \mathbf{K}_2] = 2i\mathbf{K}_1, \quad [\mathbf{K}_1, \mathbf{K}_2] = i\mathbf{K}_2.$$

However, these operators are not conserved.

4. Conformal quantum mechanics and Fick-Jacobs equation

In the last section we obtained a kind of not hermitian quantum mechanics. However, the equation (18) is related with the usual conformal quantum mechanics. In fact, using the no unitary transformation

$$C(y, t) = (b + \lambda x)^\nu \psi(y, t) \quad (24)$$

the equation (18) becomes

$$\frac{\partial \psi(y, t)}{\partial t} = -H\psi(y, t), \quad (25)$$

here

$$H = -\lambda^2 D_0 \frac{\partial^2}{\partial y^2} + \frac{g}{y^2}, \quad g = \lambda^2 D_0 \nu (\nu - 1). \quad (26)$$

Notice that H is hermitian and is the Hamiltonian for the conformal quantum mechanics (9). In fact, proposing $\psi(y, t) = e^{-Et} \Psi(y)$, we get the static Schrödinger equation for the conformal quantum mechanics

$$E\Psi(y) = H\Psi(y). \quad (27)$$

Then, the family of channels (16) is associated with the family of conformal Hamiltonians (26). Observe that for each Hamiltonian (26) we have two channels, namely each Hamiltonian is associated with two ν values. For example, $\nu = 0$ represents a cylindrical channel while $\nu = 1$ represents a conical channel, but both gave the Hamiltonian

$$H = -\lambda^2 D_0 \frac{\partial^2}{\partial y^2}. \quad (28)$$

Now, using the change of variable (17), we found that the exact solution for the Fick-Jacobs equation is given by

$$C_\nu(x, t) = B e^{-Et} (b + \lambda x)^{\frac{2\nu+1}{2}} J_{\pm(\frac{2\nu-1}{2})} \left(\pm \sqrt{\frac{E}{\lambda^2 D_0}} (b + \lambda x) \right), \quad (29)$$

here B is a constant. The parameter E can be obtained from the boundary condition. For example, if $C_\nu(L, t) = 0$, the values

$$E_n = \frac{\rho_n^2 \lambda^2 D_0}{b + \lambda L} \quad (30)$$

are obtained, where ρ_n is the n th root for the Bessel function of the order $\left(\frac{2\nu-1}{2}\right)$. In this case the initial condition $C_\nu(x, 0) = 0$ can be written as a Fourier-Bessel series.

5. AdS/Fick-Jacobs

Now, if the holographic coordinate $z \neq 0$, then the massive scalar field in AdS_{d+1} can be expressed as the equation (6), where the field $\psi(z)$ satisfies the static Schrödinger equation for the conformal quantum mechanics. Then, if the cross sectional area is given by $A(x) = (b + \lambda x)^{2\nu}$, with the mapping

$$b + \lambda x \longleftrightarrow z \quad (31)$$

$$\lambda^2 D_0 \longleftrightarrow R^{-2} = -\frac{2\Lambda}{d(d+1)}, \quad (32)$$

$$\nu(\nu - 1) \longleftrightarrow \left(\frac{d-1}{2}\right) \left(\frac{d+1}{2}\right) + m^2 R^2, \quad (33)$$

$$E \longleftrightarrow \frac{M_d^2}{R^2}, \quad (34)$$

a relation between the Fick-Jacobs equation (26) and a massive scalar field in AdS_{d+1} is obtained. Notice that the cosmological constant Λ leads the expansion of AdS_{d+1} universe, while

the diffusion coefficient D_0 leads the diffusion of particles in the channel and these both parameters are related. In addition, it can be seen that ν and d, R are related too, it is worth to mention that ν drives the geometric properties of the channel, while d and R drive the geometric properties of the AdS_{d+1} space.

In the usual AdS_{d+1}/CFT_d duality, the limit $z \rightarrow 0$ is taken [1]. However there is a correspondence with z no constant, like AdS/QCD correspondence [3]. In this paper a correspondence between the wave equation for a massive scalar field in AdS_{d+1} and Fick-Jacobs equation was gotten. Even more the axial coordinate x of the channel and the holographic coordinate z of AdS_{d+1} are identified.

6. Summary

Recently mathematical techniques developed in quantum physics have been employed to study biological systems [40]. Now, symmetries are important in quantum systems, then symmetries might be important in biological systems too. In this paper was studied the Fick-Jacobs equation which models diffusion in biological channels, zeolites and nano-channels. It was shown that, when the diffusion coefficient is a constant and is taken a particular family of channels, this equation is invariant under conformal symmetry. In addition, using the diffusion coefficient and the geometric parameters of the channels, a representation for the conformal algebra was obtained. Furthermore, it was found that for these systems the Fick-Jacobs equation is equivalent to the Schrödinger equation for the 1-dimensional conformal quantum mechanics. Moreover, using this equivalence, it was found a relation between a massive scalar field equation in AdS_{d+1} background and Fick-Jacobs equation, where the geometric parameter of the channels and the geometric parameters of AdS_{d+1} are identified.

It is well known that symmetry methods are very useful in quantum mechanics. However, in biological systems these methods are few employed. The Fick-Jacobs equation is an example where symmetries are important and useful. In a future work we will study other equations that describe biological systems using these methods.

Acknowledgments

We would like to thank P.A Horvathy for his comments about the history of Schrödinger symmetry.

- [1] J. Maldacena, *The Large N limit of Superconformal Field Theories and Supergravity*, Adv. Theor. Math. Phys. **2**, 231 (1998).
- [2] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, *Large N field theories, string theory and gravity*, Phys. Rept. **323**, 183-386 (2000).
- [3] S. J. Brodsky, G. F. de Teramond, *Light-Front Dynamics and AdS/QCD Correspondence: The Pion Form Factor in the Space- and Time-Like Regions*, Phys. Rev. D **77**, 056007 (2008).
- [4] G. F. de Teramond, Stanley J. Brodsky, *Light-Front Holography: A First Approximation to QCD*, Phys.Rev.Lett. **102** (2009) 081601.
- [5] S. A. Hartnoll, *Lectures on holographic methods for condensed matter physics*, Class. Quant. Grav. **26**, 224002 (2009) .
- [6] S. Bhattacharyya, V. E. Hubeny, S. Minwalla, M. Rangamani, *Nonlinear Fluid Dynamics from Gravity*, JHEP **0802**, 045 (2008).
- [7] M. Rangamani, *Gravity & Hydrodynamics: Lectures on the fluid-gravity correspondence*, Class. Quant. Grav. **26**, 224003 (2009).
- [8] J. de Boer, V. E. Hubeny, M. Rangamani, M. Shigemori, *Brownian motion in AdS/CFT*, JHEP **0907** 094 (2009).
- [9] W. Fischler, J. F. Pedraza, W. T. Garcia, *Holographic Brownian Motion in Magnetic Environments*, JHEP **1212**, 002 (2012).

- [10] C. R. Hagen, *Scale and conformal transformations in galilean-covariant field theory*, Phys. Rev. D **5**, 377 (1972) .
- [11] U. Niederer, *The maximal kinematical invariance group of the free Schrodinger equation*, Helv. Phys. Acta. **45**, 802 (1972).
- [12] D. T. Son, *Toward an AdS/cold atoms correspondence: a geometric realization of the Schrodinger symmetry*, Phys. Rev. D **78**, 046003 (2008).
- [13] K. Balasubramanian, J. McGreevy, *Gravity duals for non-relativistic CFTs*, Phys. Rev. Lett. **101**, 061601 (2008).
- [14] C. Duval, P. A. Horvathy, *Non-relativistic conformal symmetries and Newton-Cartan structures*, J.Phys. A **42**, 465206 (2009).
- [15] C. Duval, G. W. Gibbons. P. A. Horvathy, *Celestial mechanics, conformal structures and gravitational waves*, Phys. Rev. D **43**, 3907-3922 (1991).
- [16] C. Duval, M. Hassaine. P. A. Horvathy, *The geometry of Schrödinger symmetry in non-relativistic CFT*, Annals Phys. **324**, 1158 (2009) .
- [17] C. Leiva, M. S. Plyushchay, *Conformal symmetry of relativistic and nonrelativistic systems and AdS/CFT correspondence*, Annals Phys. **307**, 372-391 (2003).
- [18] C. Leiva, M. S. Plyushchay, *Superconformal mechanics and nonlinear supersymmetry*, JHEP **0310**, 069 (2003).
- [19] A. Anabalon, M. S. Plyushchay, *Interaction via reduction and nonlinear superconformal symmetry*, Phys. Lett. B **572**, 202-209 (2003).
- [20] A. Galajinsky, *Remark on quantum mechanics with conformal Galilean symmetry*, Phys. Rev. D **78**, 087701 (2008).
- [21] A. Galajinsky, I.V. Masterov, *Remark on quantum mechanics with $N=2$ Schrödinger supersymmetry*, Phys. Lett. B **675**, 116 (2009).
- [22] S. Lie, *Arch. Math. Nat. vid. (Kristiania)* **6**, 328 (1882).
- [23] S. Stoimenov, M. Henkel, *Dynamical symmetries of semi-linear Schrödinger and diffusion equations*, Nucl. Phys. B **723**, 205-233 (2005).
- [24] M. Jacobs, *Diffusion Processes*, Springer, New York, (1967).
- [25] D. Reguera, J. M. Rubí, *Kinetic equations for diffusion in the presence of entropic barriers*, Phys. Rev. E **64**, 061106 (2001).
- [26] D. Reguera, G. Schmid, P. S. Burada, J. M. Rubí, P. Reimann, P. Hänggi, *Entropic Transport: Kinetics, Scaling, and Control Mechanisms*, Phys. Rev. Lett. **96**, 130603 (2006).
- [27] P. S. Burada, P. Hänggi, F. Marchesoni, G. Schmid, P. Talkner, *Diffusion in Confined Geometries*, Chem. Phys. Chem. **10**, 45 (2009) .
- [28] A. M. Berezhkovskii, M. A. Pustovoit, S. M. Bezrukov, *Diffusion in a tube of varying cross section: Numerical study of reduction to effective one-dimensional description*, J. Chem. Phys. **126**, 134706 (2007) .
- [29] A. Fulinski, I. Kosinska, Z Siwy, *Transport properties of nanopores in electrolyte solutions: the diffusional model and surface currents*, New J. Phys. **7**, 132 (2005).
- [30] R. Zwanzig, *Diffusion past an entropy barrier*, J. Chem. Phys. **96**, 3926 (1992).
- [31] P. Kalinay, J. K. Percus, *Corrections to the Fick-Jacobs equation*, Phys. Rev. E **74**, 041203 (2006).
- [32] C. G. J. Jacobi, *Vorlesungen uber Dynamik* (Univ. Königsberg 1842-43). Her- ausg. A. Clebsch. Vierte Vorlesung: Das Princip der Erhaltung der lebendigen Kraft. Zweite ausg. C. G. J. Jacobis Gesammelte Werke. Supplementband. Herausg. E. Lottner. Berlin Reimer (1884).
- [33] R. Jackiw, *Introducing Scale Symmetry*, Physics Today **25**, 23 (1972).
- [34] V. de Alfaro, S. Fubini, G. Furlan, *Conformal Invariance in Quantum Mechanics*, Nuovo Cimento A **34**, 569 (1976).
- [35] R. Britto-Pacumio, J. Michelson, A. Strominger, A. Volovich, *Lectures on superconformal quantum mechanics and multiblack hole moduli spaces*, (Contributed to NATO Advanced Study Institute on Quantum Geometry) (Published in Progress in string theory and M-theory) 235-264 (1999), [hep-th/9911066].
- [36] H. E. Camblong, C. R. Ordonez, *Anomaly in conformal quantum mechanics: From molecular physics to black holes*, Phys. Rev. D **68**, 125013 (2003).
- [37] K. Sakamoto, K. Shiraishi, *Conformal Quantum Mechanics in Two Black Hole Moduli Space*, Phys. Rev. D **66** 024004(2002).
- [38] C. Chamon, R. Jackiw R, S-Y. Pi, L. Santos, *Conformal quantum mechanics as the CFT_1 dual to AdS_2* , Phys. Lett. B **701**, 503 (2011) .
- [39] R. Jackiw, S-Y. Pi S, *Conformal Blocks for the 4-Point Function in Conformal Quantum Mechanics*, Phys. Rev. D **86**, 045017, Erratum-ibid. D **86** 089905 (2012).
- [40] L. Abbott, E. Farhi, S. Gutmann, *The path integral for dendritic trees*, Biol. Cybern. **66** 49 (1991).