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Conformal symmetry in quantum finance

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Abstract. The quantum finance symmetries are studied. In order to do this, the one dimensional free non-relativistic particle and its symmetries are revisited and the particle mass is identified as the inverse of square of the volatility. Furthermore, using financial variables, a Schrödinger algebra representation is constructed. In addition, it is shown that the operators of this last representation are not hermitian and not conserved.

1. Introduction

Lately, mathematical techniques developed in physics have been employed to study systems from other areas. For example, the Black-Scholes-Merton equation [1, 2] is important in theoretical finance and it can be mapped to the one dimensional free Schrödinger equation [3]. Then, mathematical techniques that arise in quantum mechanics can be used to study financial phenomena, this fact allowed the birth of a new discipline, the so call Quantum Finance [3]. Now, it is well known that when symmetries are present in a physical system we can get at the properties of a system without completely solving all the equations that describe the system, in fact symmetries imply conserved quantities. The conformal symmetry is important in physics, for example the relativistic conformal group is the largest symmetry group of special relativity [4]. In addition, the free Schrödinger equation is invariant under the Schrödinger group, which is a non-relativistic conformal group [5, 6]. It is worth mentioning that in 1882 Sophus Lie showed that the Fick equation, which describes diffusion, is invariant under the Schrödinger group [7].

In this paper, it will be shown that the Black-Scholes-Merton equation is invariant under the Schrödinger group. In order to do this, the one dimensional free non-relativistic particle and its symmetries will be revisited. To get the Black-Scholes-Merton equation symmetries, the particle mass is identified as the inverse of square of the volatility. Furthermore, using financial variables, a Schrödinger algebra representation is constructed. In addition, it is shown that the operators of this last representation are not hermitian and not conserved.

This paper is organized as follows: in section 2 we provide a brief overview of the one dimensional non-relativistic free particle and its symmetries; in section 3 the one dimensional free Schrödinger equation and its symmetries are studied; in section 4 the Black-Scholes-Merton equation and its symmetries are studied. Finally, in section 5 a summary is given.

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2. Free particle

The action for one-dimensional non-relativistic free particle is given by

$$S = \int dt \frac{m}{2} \left(\frac{dx}{dt}\right)^2, \tag{1}$$

this is the simplest mechanical system. Now, using the coordinates transformation

$$t' = \frac{\alpha t + \beta}{\gamma t + \delta}, \qquad x' = \frac{ax + vt + c}{\gamma t + \delta}, \qquad a^2 = \alpha\delta - \beta\gamma \neq 0, \tag{2}$$

where $\alpha, \beta, \gamma, \delta, a, v, c$ are constants, the action (1) transforms as

$$S' = \int dt' \frac{m}{2} \left(\frac{dx'}{dt'}\right)^2 = S + \frac{m}{2} \int dt \left(\frac{d\phi(x,t)}{dt}\right), \tag{3}$$

where

$$\phi(x,t) = \frac{1}{a^2} \left(2avx + v^2 t - \frac{\gamma\,(ax + vt + c)^2}{\gamma t + \delta}\right). \tag{4}$$

Then, the equation of motion for one dimensional free particle is invariant under the conformal coordinate transformations (2). This coordinates transformation includes temporal translations

$$t' = t + \beta, \qquad x' = x, \tag{5}$$

spatial translations

$$t', \qquad x' = x + c, \tag{6}$$

Galileo‘s transformations

$$t', \qquad x' = x + vt, \tag{7}$$

anisotropic scaling

$$t' = a^2 t, \qquad x' = ax \tag{8}$$

and the special conformal transformations

$$t' = \frac{1}{\gamma t + 1}, \qquad x' = \frac{x}{\gamma t + 1}. \tag{9}$$

It is shown below that the conformal coordinate transformations (2) and the quantity (4) are useful to study Black-Scholes-Merton equation symmetries.

2.1. Conservative quantities

For the one dimensional non-relativistic particle, the following quantities

$$\begin{aligned} P &= m\dot{x}, && (10)\\ H &= \frac{P^2}{2m}, && (11)\\ G &= tP - mx, && (12)\\ K_1 &= tH - \frac{1}{2}xP, && (13)\\ K_2 &= t^2 H - txP + \frac{m}{2}x^2 && (14) \end{aligned}$$

are conserved.

The momentum P is associated with spatial translations (6). The Hamiltonian H is associated with temporal translations (5) . The quantity G is associated with Galileo's transformations (7). While K_1 is associated with anisotropic scaling (8) and K_2 is associated with the special conformal transformations (9).

Furthermore, using the Poisson brackets, it can be shown that the following relations

$$\{P,H\} = 0, \tag{15}$$
$$\{P,K_1\} = \frac{1}{2}P, \tag{16}$$
$$\{P,K_2\} = G, \tag{17}$$
$$\{P,G\} = m, \tag{18}$$
$$\{H,K_1\} = H, \tag{19}$$
$$\{H,G\} = P, \tag{20}$$
$$\{H,K_2\} = 2K_1, \tag{21}$$
$$\{K_1,K_2\} = K_2, \tag{22}$$
$$\{K_1,G\} = \frac{1}{2}G, \tag{23}$$
$$\{K_2,G\} = 0 \tag{24}$$

are satisfied.

2.2. Schrödinger group

In quantum mechanics, if a particle is in x_0 at time t_0, the amplitude to travel to x in a time $T = t - t_0$ is given by

$$U(x,t;x_0,t_0) = \left(\frac{m}{2\pi i(t-t_0)}\right)^{\frac{1}{2}} e^{\frac{i}{\hbar}\int_{t_0}^{t} d\tilde{t}\frac{m}{2}\left(\frac{dx}{d\tilde{t}}\right)^2}, \tag{25}$$

which satisfies the Schrödinger equation

$$i\hbar\frac{\partial U(x,t;x_0,t_0)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 U(x,t;x_0,t_0)}{\partial x^2}. \tag{26}$$

Now, in another system with coordinates x', t', x'_0, t'_0 we have the amplitude

$$U'(x',t';x'_0,t'_0) = \left(\frac{m}{2\pi i(t'-t'_0)}\right)^{\frac{1}{2}} e^{\frac{i}{\hbar}\int_{t'_0}^{t'} d\tilde{t}'\frac{m}{2}\left(\frac{dx'}{d\tilde{t}'}\right)^2}, \tag{27}$$

which satisfies the Schrödinger equation (26) on primed coordinates (x', t'). In addition, using the conformal transformations (2) we have

$$t' = \frac{\alpha t + \beta}{\gamma t + \delta}, \qquad x' = \frac{ax + vt + c}{\gamma t + \delta}, \tag{28}$$
$$t'_0 = \frac{\alpha t_0 + \beta}{\gamma t_0 + \delta}, \qquad x'_0 = \frac{ax_0 + vt_0 + c}{\gamma t_0 + \delta}. \tag{29}$$

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Notice that, without loss of generality, we can take $x_0 = 0$ and $t_0 = 0$, in this case

$$t' - t'_0 = \frac{a^2 t}{\delta(\gamma t + \delta)}. \tag{30}$$

Then, using the equation (3), we have

$$U'(x', t'; x'_0, t'_0) = \sqrt{\frac{\delta}{a^2}} \left(\sqrt{\gamma t + \delta}\right) e^{\frac{im}{2\hbar}\phi(x,t)} U(x, t; 0, 0), \tag{31}$$

where $\phi(x,t)$ is given by (4). Now, due that $U(x, t; x_0, t_0)$ is solution for the Schrödinger equation (26) on coordinates (x, t), and $U'(x', t'; x'_0, t'_0)$ satisfies the same equation on primed coordinates (x', t'), the expression (31) implies that the Schrödinger equation for the one dimensional non-relativistic free particle

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2} \tag{32}$$

is invariant under the conformal coordinate transformations (2), where the wave function transforms as

$$\psi'(x', t') = \left(\sqrt{\gamma t + \delta}\right) e^{\frac{im}{2\hbar}\phi(x,t)} \psi(x, t), \tag{33}$$

and $\phi(x,t)$ is given by (4).

Using other methods, the conformal symmetry for the free Schrödinger equation was found by Niederer and Hagen in 1972 [6, 5]. However, this symmetry was obtained by S. Lie in 1882 while he was studying the Fick equation [7].

Furthermore, according to quantum mechanics, the quantities (10)-(14) are represented by the operators

$$\hat{P} = -i\hbar \frac{\partial}{\partial x}, \tag{34}$$

$$\hat{H} = \frac{\hat{P}^2}{2m}, \tag{35}$$

$$\hat{G} = t\hat{P} - mx, \tag{36}$$

$$\hat{K}_1 = t\hat{H} - \frac{1}{4}\left(x\hat{P} + \hat{P}x\right), \tag{37}$$

$$\hat{K}_2 = t^2\hat{H} - \frac{t}{2}\left(x\hat{P} + \hat{P}x\right) + \frac{m}{2}x^2. \tag{38}$$

These operators satisfy the Schrödinger algebra

$$\left[\hat{P}, \hat{H}\right] = 0, \tag{39}$$

$$\left[\hat{P}, \hat{K}_1\right] = \frac{i\hbar}{2}\hat{P}, \tag{40}$$

$$\left[\hat{P}, \hat{K}_2\right] = i\hbar\hat{G}, \tag{41}$$

$$\left[\hat{P}, \hat{G}\right] = i\hbar m, \tag{42}$$

$$\left[\hat{H}, \hat{K}_1\right] = i\hbar \hat{H}, \tag{43}$$

$$\left[\hat{H}, \hat{G}\right] = i\hbar \hat{P}, \tag{44}$$

$$\left[\hat{H}, \hat{K}_2\right] = 2i\hbar \hat{K}_1, \tag{45}$$

$$\left[\hat{K}_1, \hat{K}_2\right] = i\hbar \hat{K}_2, \tag{46}$$

$$\left[\hat{K}_1, \hat{G}\right] = \frac{i\hbar}{2} \hat{G}, \tag{47}$$

$$\left[\hat{K}_2, \hat{G}\right] = 0, \tag{48}$$

which is similar to the algebra (15)-(24). It is possible to show that the operators (34)-(38) are conserved.

In the next section, it will be shown that Black-Scholes-Merton equation is invariant under Schrödinger symmetry.

3. The Black-Scholes-Merton equation and the Black-Scholes Formula

Today, an esential question in financial world refers to how obtain the price of an option. Financial markets use the Black-Scholes option pricing model and it is the basis of sophisticated methods of options valuation. A remarkable result in this field is given by the Black-Scholes-Merton equation [1, 2]

$$\frac{\partial C(s,t)}{\partial t} = -\frac{\sigma^2}{2} s^2 \frac{\partial^2 C(s,t)}{\partial s^2} - rs\frac{\partial C(s,t)}{\partial s} + rC(s,t), \tag{49}$$

where C is the price of an option, s is the price of the stock, σ is the volatility and r is the annualized risk-free interest rate. The Black-Scholes-Merton equation have to be solved with condition

$$C(s,T) = (s-K)\theta(s-K), \tag{50}$$

where K is the strike price of an option, T is the time to expiration and $\theta(x)$ is the Heaviside function.

Amazingly, the Black-Scholes-Merton equation (49) is equivalent to the Schrödinger equation [3]. In fact, using the change of variable

$$s = e^x \tag{51}$$

in the equation (49), the following result

$$\frac{\partial C(x,t)}{\partial t} = -\frac{\sigma^2}{2}\frac{\partial^2 C(x,t)}{\partial x^2} + \left(\frac{\sigma^2}{2} - r\right)\frac{\partial C(x,t)}{\partial x} + rC(x,t). \tag{52}$$

is gotten. Additionally, if

$$C(x,t) = e^{\left[\frac{1}{\sigma^2}\left(\frac{\sigma^2}{2}-r\right)x + \frac{1}{2\sigma^2}\left(\frac{\sigma^2}{2}+r\right)^2 t\right]}\Psi(x,t) \tag{53}$$

the following equation

$$\frac{\partial \Psi(x,t)}{\partial t} = -\frac{\sigma^2}{2}\frac{\partial^2 \Psi(x,t)}{\partial x^2} \tag{54}$$

is obtained, which is Schrödinger-like wave equation (32). Notice that $1/\sigma^2$ has the role of particle mass m. In addition, with the change of variable $\tau = T - t$, we get

$$\frac{\partial \Psi(x,\tau)}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 \Psi(x,\tau)}{\partial x^2}, \tag{55}$$

which is the heat equation.

Now, using the solution for the free Schrödinger equation, we can get the solution for the Black-Sholes-Merton equation. In fact, using the initial condition (50) we get

$$C(s,t) = sN(d_+) - Ke^{-r(T-t)}N(d_-) \tag{56}$$

where

$$N(z) = \int_{-\infty}^{z} du \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}, \quad d_\pm = \frac{\ln\left(\frac{s}{K}\right) + (T-t)\left(r \pm \frac{\sigma^2}{2}\right)}{\sigma\sqrt{T-t}}.$$

The equation (56) is the so call Black-Scholes formula and it can be obtained using quantum mechanics techniques.

4. The Schrödinger group and the Black-Scholes-Merton equation

Due that the Schrödinger equation (32) is invariant under conformal transformation (2), the equation (55) is invariant under the same transformation. In this case the function $\psi(x,t)$ transforms as

$$\Psi'\left(x',t'\right) = \left(\sqrt{\gamma t + \delta}\right) e^{\frac{1}{2\sigma^2}\phi(x,t)}\Psi(x,t), \tag{57}$$

where $\phi(x,t)$ is given by (4). Notice that the particle mass m is changed for $1/\sigma^2$.

Using the change of variable (51), the coordinate transformations (2) can be written as

$$t' = \frac{\alpha t + \beta}{\gamma t + \delta}, \qquad s' = e^{\left(\frac{vt+c}{\gamma t+\delta}\right)} s^{\left(\frac{a}{\gamma t+\delta}\right)}. \tag{58}$$

Through a long but straightforward calculation, it can be shown that the Black-Scholes-Merton equation (49) is invariant under this last transformation, where the price $C(s,t)$ transforms as

$$C'\left(s',t'\right) \;=\; \left(\sqrt{\gamma t + \delta}\right) s^{\Phi_1(s,t)} e^{\Phi_2(s,t)} C(s,t), \tag{59}$$

here

$$\Phi_1(s,t) \;=\; \frac{-2a^2\gamma\left(\frac{\sigma^2}{2} - r\right)t + 2a\left(v\delta - \gamma c\right) + 2a^2(a-\delta)\left(\frac{\sigma^2}{2} - r\right)}{2a^2\sigma^2\left(\gamma t + \delta\right)} - \frac{\gamma a^2\left(\ln s\right)}{2a^2\sigma^2\left(\gamma t + \delta\right)}$$

and

$$\Phi_2(s,t) = \left(\frac{a^2\left(\frac{\sigma^2}{2}+r\right)^2(\alpha-\delta)+2a^2v\left(\frac{\sigma^2}{2}-r\right)+v\,(v\delta-2\gamma c)}{2\sigma^2a^2(\gamma t+\delta)}\right)t$$
$$+\frac{a^2\beta\left(\frac{\sigma^2}{2}+r\right)^2+2a^2\left(\frac{\sigma^2}{2}-r\right)c-\gamma c^2-\gamma a^2\left(\frac{\sigma^2}{2}+r\right)^2t^2}{2\sigma^2a^2(\gamma t+\delta)}.$$

Now, the Black-Scholes-Merton equation (49) can be written as

$$\frac{\partial C(s,t)}{\partial t} = \hat{\mathbf{H}}C(s,t), \tag{60}$$

where

$$\hat{\mathbf{H}} = -\frac{\sigma^2}{2}s^2\frac{\partial^2}{\partial s^2} - rs\frac{\partial}{\partial s} + r. \tag{61}$$

Moreover, using the operator

$$\hat{\Pi} = -is\frac{\partial}{\partial s} + \frac{i}{\sigma^2}\left(\frac{\sigma^2}{2}-r\right), \tag{62}$$

the operator $\hat{\mathbf{H}}$ can be rewritten as

$$\hat{\mathbf{H}} = \frac{\sigma^2}{2}\hat{\Pi}^2 + \frac{1}{2\sigma^2}\left(\frac{\sigma^2}{2}+r\right)^2. \tag{63}$$

Notice that the operator $\hat{\mathbf{H}}$ is similar to the Hamiltonian operator $\hat{H}$ (35), where the particle mass m is associated with $1/\sigma^2$. However, the operator $\hat{\mathbf{H}}$ is not hermitian.

Additionally, using the operator (62) it is possible construct quantities related with the non-relativistic free particle conserved quantities (34)-(38). In fact, the operators

$$\hat{\Pi} = -is\frac{\partial}{\partial s} + \frac{i}{\sigma^2}\left(\frac{\sigma^2}{2}-r\right), \tag{64}$$

$$\hat{\mathbf{H}}_0 = \frac{\sigma^2}{2}\hat{\Pi}^2, \tag{65}$$

$$\hat{\mathbf{G}} = t\hat{\Pi} - \frac{1}{\sigma^2}\ln s, \tag{66}$$

$$\hat{\mathbf{K}}_1 = t\hat{\mathbf{H}}_0 - \frac{1}{4}\left(\ln s\hat{\Pi} + \hat{\Pi}\ln s\right), \tag{67}$$

$$\hat{\mathbf{K}}_2 = t^2\hat{\mathbf{H}}_0 - \frac{t}{2}\left(\ln s\hat{\Pi} + \hat{\Pi}\ln s\right) + \frac{1}{2\sigma^2}(\ln s)^2 \tag{68}$$

can be constructed, which are similar to the quantities (34)-(38). Now, using the relation

$$[\ln s, \hat{\Pi}] = i, \tag{69}$$

the algebra

$$\left[\hat{\Pi}, \hat{\mathbf{H}}_0\right] = 0, \quad (70)$$

$$\left[\hat{\Pi}, \hat{\mathbf{K}}_1\right] = \frac{i}{2}\hat{\Pi}, \quad (71)$$

$$\left[\hat{\Pi}, \hat{\mathbf{K}}_2\right] = i\hat{\mathbf{G}}, \quad (72)$$

$$\left[\hat{\Pi}, \hat{\mathbf{G}}\right] = \frac{i}{\sigma^2}, \quad (73)$$

$$\left[\hat{\mathbf{H}}_0, \hat{\mathbf{K}}_1\right] = i\hat{\mathbf{H}}_0 \quad (74)$$

$$\left[\hat{\mathbf{H}}_0, \hat{\mathbf{G}}\right] = i\hat{\Pi}, \quad (75)$$

$$\left[\hat{\mathbf{H}}_0, \hat{\mathbf{K}}_2\right] = 2i\hat{\mathbf{K}}_1, \quad (76)$$

$$\left[\hat{\mathbf{K}}_1, \hat{\mathbf{K}}_2\right] = i\hat{\mathbf{K}}_2, \quad (77)$$

$$\left[\hat{\mathbf{K}}_1, \hat{\mathbf{G}}\right] = \frac{i}{2}\hat{\mathbf{G}}, \quad (78)$$

$$\left[\hat{\mathbf{K}}_2, \hat{\mathbf{G}}\right] = 0. \quad (79)$$

is satisfied. Then the operators (64)-(68) satisfy the Schrödinger algebra. However, the operators (64)-(68) are not hermitian and are not conserved. Then, the equivalence between the Black-Scholes-Merton and the free Schrödinger equation is not exactly, this last happen because the transformation (53) is not unitary.

Using other methods, the Black-Scholes-Merton symmetries have been studied in [8].

5. Summary

It was shown that the Black-Scholes-Merton equation is invariant under the Schrödinger group. In order to do this, the one dimensional free non-relativistic particle and its symmetries were revisited. To get the Black-Scholes-Merton equation symmetries, the particle mass was identified as the inverse of square of the volatility. Besides, using financial variables, a Schrödinger algebra representation was constructed. However, the operators of this last representation are not hermitian and not conserved.

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