Electrodynamics à la Hořava

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We study an electrodynamics consistent with anisotropic transformations of space-time with an arbitrary dynamic exponent z. The equations of motion and conserved quantities are explicitly obtained. We show that the propagator of this theory can be regarded as a quantum correction to the usual propagator. Moreover we obtain that both the momentum and angular momentum are not modified, but their conservation laws do change. We also show that in this theory the speed of light and the electric charge are modified with z. The magnetic monopole in this electrodynamics and its duality transformations are also investigated. For that we found that there exists a dual electrodynamics, with higher derivatives in the electric field, invariant under the same anisotropic transformations.

1. Introduction

In several areas of physics the scale-invariant systems are important. These systems have been recognized in the context of condensed matter [1,2] and theories of modified gravity such as MOND [3], see also [4]. Of particular importance are the systems invariant under anisotropic scale-transformations of the space-time of the form

$$t \to b^z t, \qquad \vec{x} \to b\vec{x}, \tag{1}$$

where z plays the role of a dynamical critical-exponent. These transformations arise in the non-relativistic limit of string theory [5,6] and, through the duality AdS/CFT, this theory can be related to condensed matter systems [7,8]. Recently, in addition, P. Hořava proposed a gravity compatible with (1). This gravity is non-relativistic but at large distances retrieves a gravity similar to Einstein's [9]. A noteworthy fact of Hořava gravity is that replaces the usual dispersion relation by

$$k_0^2 - G(k^2)^z = 0, \qquad G = \text{constant}, k^2 = k_1^2 + k_2^2 + k_3^2, \tag{2}$$

but more importantly, for the case of $z = 3$, yields a ghost-free gravity which is renormalizable by power counting. This new gravity has features that makes it notable; for instance, in a natural way has an alternative mechanism to inflation [10]. In addition, it can explain some cosmological phenomena without introducing dark matter [11]; some systems in this gravity can be found in [12,13]. Another feature is that the dispersion relation (2) arises not from a geodesic equation but from a mechanics invariant under (1), see [14,15,16,17,18,19]. Hořava gravity is recent and has several aspects not well understood; for example, it presents some dynamical problems [20,21,22,23,24,25]; see also [26]. A modification to Hořava gravity, free from these problems, can be found in [27,28,29]; however see [30]. From all this, Hořava gravity is without a doubt an important step towards understanding the quantum aspects of gravity.

Field theories consistent with the dispersion relation (2) for the case of $z = 2$ have been studied [31,32,33,34], some systems with $z = 3$ can be seen in [35,36]. Systems with arbitrary z have been studied in: black holes [37], nonrelativistic AdS/CFT duality [38,39], and the causal structure of Schrödinger space [40]. However, the case of z arbitrary has been seldom considered in field theory. In this paper, based on the initial works [41,42] and [43], we study an electrodynamics compatible with the transformations (1) for arbitrary z. Modified Maxwell's equations and their conserved quantities are first obtained for the case of $z = 2$ and then, based on these results, a general model that includes derivatives of higher order in the magnetic field is constructed. For this model we obtain a dispersion relation of the form

$$\frac{\omega^2 \alpha}{c^2} = \sum_{n \geq 1} b_n \left(k^2 \right)^n , \tag{3}$$

and show that α acts as a refraction index. Notice that the dispersion relation (2) is a particular case of (3). The conserved quantities are also obtained; in particular it is shown that energy is modified, but the momentum and angular momentum remain unchanged. In addition, we show the existence of a conserved quantity additional to the usual ones. For this system we propose the modified Maxwell's equations with sources and obtain their conservation laws. In particular, the modifications to the stress tensor are provided. Moreover, from duality transformations we consider the problem of the magnetic monopole and show the existence of a dual electrodynamics compatible with the dispersion relation (3). Finally, we show that the propagator of this theory can be regarded as a quantum correction to the usual Maxwell's propagator.

It is worth mentioning that non-relativistic Maxwell theory has been already studied in another context [44,45,46]. Other models with higher derivatives in the magnetic fields can be found in [47,48,49].

This paper is organized as follows: in section 2 Hořava electrodynamics for the

case of $z = 2$ is studied; in the section 3 we present the general case of arbitrary z; an analysis of the magnetic monopole question appears in section 4; we obtain the Green function in section 5; finally our results are summarized in section 6.

2. Electrodynamics à la Hořava: $z = 2$

In this section we study Hořava electrodynamics for the case of $z = 2$ originally proposed in [42]. For completeness we recall the definitions

$$E_i = -\left(\partial_{ct}A_i + \partial_i\phi\right), \quad B_i = \left(\vec{\nabla} \times \vec{A}\right)_i, \quad F_{ij} = \partial_i A_j - \partial_j A_i = \epsilon_{ijk}B_k. \quad (4)$$

Now, let us assume we have the action

$$S = \int cdt d\vec{x} \mathcal{L} = \int cdt d\vec{x} \left(\alpha E_i E_i + \beta \partial_i F_{ik} \partial_j F_{jk}\right). \quad (5)$$

By requesting $\delta S = 0$ one finds the equations of motion

$$\alpha \partial_{ct} E_k + \beta \partial^2 \partial_i F_{ik} = 0, \qquad \partial_i E_i = 0. \quad (6)$$

Consideration of (4) leads to the modified Maxwell's equations

$$\vec{\nabla} \cdot \vec{E} = 0, \quad (7)$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}, \quad (8)$$

$$\vec{\nabla} \cdot \vec{B} = 0, \quad (9)$$

$$\vec{\nabla} \times \left(\beta \nabla^2 \vec{B}\right) = \frac{\alpha}{c}\frac{\partial \vec{E}}{\partial t}. \quad (10)$$

From (8) and (10) we obtain

$$\left(\beta \left(\nabla^2\right)^2 - \frac{\alpha}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0, \qquad \left(\beta \left(\nabla^2\right)^2 - \frac{\alpha}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{B} = 0. \quad (11)$$

Then, the dispersion relation for plane waves is $\beta \left(k^2\right)^2 = -\alpha\omega^2/c^2$.

The equations (7)-(10) are invariant under the transformations of scale

$$\vec{E} \to b^{-3}\vec{E}, \quad \vec{B} \to b^{-2}\vec{B}, \quad t \to b^2 t, \quad \vec{x} \to b\vec{x}, \quad (12)$$

however, the action transforms as

$$S \to b^{-1}S. \quad (13)$$

A similar case occurs to the harmonic oscillator, whose equations of motion are invariant under the scale transformations $x \to bx$, but the action does not have this invariance.

2.1. *Conserved quantities*

By considering the equations of motion (7)-(10) one may show that the quantities

$$\mathcal{E} = \frac{c}{8\pi} \int d\vec{x} \left(\alpha \vec{E}^2 + \beta \vec{B} \cdot \nabla^2 \vec{B} \right), \tag{14}$$

$$\vec{P} = \frac{1}{4\pi c} \int d\vec{x} \left(\alpha \vec{E} \times \vec{B} \right), \tag{15}$$

$$\vec{L} = \frac{1}{4\pi c} \int d\vec{x} \left[\vec{x} \times \left(\alpha \vec{E} \times \vec{B} \right) \right], \tag{16}$$

are conserved. From these we see that the energy is different to the usual one, but both the momentum and angular momentum remain unchanged. Below we will find a generalization of these quantities and their conservation laws for arbitrary z.

Although Noether's theorem does not give a conserved charge for symmetry (12), in the next section we will associate a conserved charge with this.

3. Electrodynamics à la Hořava: General Case

We now move on to propose a model that allows one to obtain a dispersion relation of the form (2) valid for arbitrary z. By assuming $f(x)$ a smooth function, we pose the action

$$S = \int cdtd\vec{x}\mathcal{L} = \int cdtd\vec{x} \left(\alpha E_i E_i - \vec{B} \cdot f\left(\nabla^2\right) \vec{B} \right)$$

$$= \int cdtd\vec{x} \left(\alpha E_i E_i - \frac{1}{2} F_{ij} f\left(\nabla^2\right) F_{ij} \right). \tag{17}$$

First let us observe that in units of momenta we have

$$[cdtdx^3] = -(3+z), \quad [E_i] = (z+1), \quad [B_i] = 2, \quad [\alpha] = -(z-1). \tag{18}$$

Now, if $f(x) = \sum_{n\geq 1} a_n x^{n-1}$ and $[a_n] = z - 1 - 2(n-1)$, then $\left[f\left(\nabla^2\right) \right] = z - 1$; therefore, S has no units.

Note us that the Lagrangian can be regarded as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I, \tag{19}$$

where,

$$\mathcal{L}_0 = \alpha \vec{E} \cdot \vec{E} - a_1 \vec{B} \cdot \vec{B} \tag{20}$$

the free part of the Lagrangian and

$$\mathcal{L}_I = \sum_{n=2}^{\infty} a_n \vec{B} \cdot \left(\nabla^2\right)^{n-1} \vec{B}, \tag{21}$$

the interaction one.

Before performing the variation of S, notice that carrying out n integrations by parts and neglecting the boundary terms we obtain

$$\int d\vec{x} F_{ij} \left(\nabla^2\right)^n \delta F_{ij} = (-)^{2n} \int d\vec{x} \delta F_{ij} \left(\nabla^2\right)^n F_{ij}, \tag{22}$$

so that

$$\int d\vec{x} \delta \left(F_{ij} \left(\nabla^2\right)^n F_{ij} \right) = \int d\vec{x} \left(\delta F_{ij} \left(\nabla^2\right)^n F_{ij} + F_{ij} \left(\nabla^2\right)^n \delta F_{ij} \right)$$

$$= 2 \int d\vec{x} \delta F_{ij} \left(\nabla^2\right)^n F_{ij}, \tag{23}$$

from where we may define

$$\int d\vec{x} F_{ij} f \left(\nabla^2\right) F_{ij}. \tag{24}$$

Thus

$$\delta \int d\vec{x} F_{ij} f \left(\nabla^2\right) F_{ij} = \int d\vec{x} \delta \left(F_{ij} f \left(\nabla^2\right) \right) F_{ij}$$

$$= \int d\vec{x} \delta \left(F_{ij} \left[\sum_{n \geq 1} a_n \left(\nabla^2\right)^{n-1} \right] F_{ij} \right) = \sum_{n \geq 1} a_n \int d\vec{x} \delta \left(F_{ij} \left(\nabla^2\right)^{n-1} F_{ij} \right)$$

$$= 2 \sum_{n \geq 1} a_n \int d\vec{x} \delta F_{ij} \left(\nabla^2\right)^{n-1} F_{ij} = 2 \int d\vec{x} \delta F_{ij} f \left(\nabla^2\right) F_{ij}. \tag{25}$$

By using definition (4) and neglecting boundary terms one gets to

$$\delta \int d\vec{x} F_{ij} f \left(\nabla^2\right) F_{ij} = -4 \int d\vec{x} \delta A_j \partial_i \left(f \left(\nabla^2\right) F_{ij} \right). \tag{26}$$

Taking these results into account and once again neglecting boundary terms one obtains

$$\delta S = \int cdt d\vec{x} 2 \Big[\delta A_i \big(f(\nabla^2) \partial_j F_{ji} + \alpha \partial_{ct} E_i \big) + \alpha \delta \phi \partial_i E_i \Big]. \tag{27}$$

Therefore, $\delta S = 0$ implies the equations of motion

$$\vec{\nabla} \cdot \vec{E} = 0, \tag{28}$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}, \tag{29}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{30}$$

$$\vec{\nabla} \times \left(f \left(\nabla^2\right) \vec{B} \right) = \frac{\alpha}{c} \frac{\partial \vec{E}}{\partial t}. \tag{31}$$

For $f(x) = a_z x^{z-1}$ these yield

$$\vec{\nabla} \cdot \vec{E} = 0, \qquad \vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}, \tag{32}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \qquad \vec{\nabla} \times \left(a_z \left(\nabla^2\right)^{z-1} \vec{B} \right) = \frac{\alpha}{c} \frac{\partial \vec{E}}{\partial t}. \tag{33}$$

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Notice that the dimensions of α change for each z. As $[\alpha] = -(z-1)$, we may assume that α is of the form l^{z-1}, where l is a constant with units of length. Under this assumption, it is clear that $\alpha = 1$ for $z = 1$, but $\alpha \neq 1$ for $z \neq 1$. Therefore, the constant α changes with z.

In addition to Eqs. (29) and (31) we obtain the modified wave equations

$$\left(f\left(\nabla^2\right)\nabla^2 - \frac{\alpha}{c^2}\frac{\partial^2}{\partial t^2} \right)\vec{E} = 0, \tag{34}$$

$$\left(f\left(\nabla^2\right)\nabla^2 - \frac{\alpha}{c^2}\frac{\partial^2}{\partial t^2} \right)\vec{B} = 0. \tag{35}$$

For the case of plane waves one obtains the dispersion relation

$$\frac{\omega^2\alpha}{c^2} = f(-k^2)k^2 = \sum_{n\geq 1}(-)^{n-1}a_n\left(k^2\right)^n. \tag{36}$$

In particular, if $a_n = 0$ for $n \neq z$ and $a_z = (-)^{z-1}G$, this yield

$$\frac{\omega^2\alpha}{c^2} = G(k^2)^z, \tag{37}$$

which is Hořava dispersion relation (2). By defining $c' = c/n$ with $n = \sqrt{\alpha}$, we can consider α as a refraction index that changes with z.

3.1. *Case with sources*

The modified Maxwell's equations with sources are

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho, \tag{38}$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}, \tag{39}$$

$$\vec{\nabla}\cdot\vec{B} = 0, \tag{40}$$

$$\vec{\nabla}\times\left(f\left(\nabla^2\right)\vec{B}\right) = \frac{4\pi}{c}\vec{J} + \frac{\alpha}{c}\frac{\partial\vec{E}}{\partial t}. \tag{41}$$

By considering equations (38) and (41) we obtain conservation of the electric charge:

$$\frac{\partial\rho'}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0, \tag{42}$$

with $\rho' = \alpha\rho$. This implies that the electric charge is modified by z. For instance, for a point particle the electric charge changes from e to $e' = \alpha e$. Below we will find this phenomenon also present in the Lorentz force.

3.2. *Conserved quantities*

For $\rho = 0$ and $\vec{J} = \vec{0}$ one obtains the conserved quantities

$$\mathcal{E} = \frac{c}{8\pi} \int d\vec{x} \left(\alpha \vec{E} \cdot \vec{E} + \vec{B} \cdot f \left(\nabla^2 \right) \vec{B} \right), \tag{43}$$

$$\vec{P} = \frac{1}{4\pi c} \int d\vec{x} \left(\alpha \vec{E} \times \vec{B} \right), \tag{44}$$

$$\vec{L} = \frac{1}{4\pi c} \int d\vec{x} \left[\vec{x} \times \left(\alpha \vec{E} \times \vec{B} \right) \right]. \tag{45}$$

As it can be seen, the energy is modified but the momentum and angular momentum remain unchanged.

For the general case it is valid that

$$\frac{1}{c} \frac{d\mathcal{E}}{dt} = - \int d\vec{x} \vec{E} \cdot \vec{J} - \frac{c}{4\pi} \oint da \left(\vec{E} \times f \left(\nabla^2 \right) \vec{B} \right) \cdot \hat{n}$$
$$+ \frac{1}{8\pi} \int d\vec{x} \left(\vec{B} \cdot f \left(\nabla^2 \right) \frac{\partial \vec{B}}{\partial t} - \frac{\partial \vec{B}}{\partial t} \cdot f \left(\nabla^2 \right) \vec{B} \right). \tag{46}$$

The last integral in this expression is a boundary term. In fact, carrying out a perturbative expansion we find

$$\frac{1}{8\pi} \int d\vec{x} \left(\vec{B} \cdot f \left(\nabla^2 \right) \frac{\partial \vec{B}}{\partial t} - \frac{\partial \vec{B}}{\partial t} \cdot f \left(\nabla^2 \right) \vec{B} \right) = - \frac{c}{4\pi} \oint da u_l n_l, \tag{47}$$

with

$$u_l = \frac{1}{2c} \Big[a_2 \left(\partial_t B_i \partial_l B_i - B_i \partial_l \partial_t B_i \right) + a_3 \left(\partial_t B_i \nabla^2 \partial_l B_i - B_i \nabla^2 \partial_t \partial_l B_i \right. $$
$$\left. + \partial_m B_i \partial_m \partial_l \partial_t B_i - \partial_t \partial_m B_i \partial_m \partial_m B_i \right) + \cdots \Big]. \tag{48}$$

Hence,

$$\frac{1}{c} \frac{d\mathcal{E}}{dt} = - \int d\vec{x} \vec{E} \cdot \vec{J} - \frac{c}{4\pi} \oint da \left(\vec{E} \times f \left(\nabla^2 \right) \vec{B} + \vec{u} \right) \cdot \hat{n}. \tag{49}$$

Notice that in this case the energy flux is not directly related to the momentum $\vec{P}$.

One also finds that

$$\frac{dP_i}{dt} = - \int d\vec{x} \left(\alpha \rho \vec{E} + \frac{\vec{J} \times \vec{B}}{c} \right)_i + \frac{1}{4\pi} \oint da \tilde{\tau}_{ij} n_j$$
$$+ \frac{1}{8\pi} \int d\vec{x} \left[\partial_i \vec{B} \cdot f \left(\nabla^2 \right) \vec{B} - \vec{B} \cdot f \left(\nabla^2 \right) \partial_i \vec{B} \right], \tag{50}$$

with

$$\tilde{\tau}_{ij} = \alpha E_i E_j + B_j f \left(\nabla^2 \right) B_i - \frac{\delta_{ij}}{2} \left(\alpha \vec{E} \cdot \vec{E} + \vec{B} \cdot f \left(\nabla^2 \right) \vec{B} \right). \tag{51}$$

The last term in (50) is a boundary integral as

$$
\frac{1}{8\pi c} \int d\vec{x} \left[\partial_i \vec{B} \cdot f\left(\nabla^2\right) \vec{B} - \vec{B} \cdot f\left(\nabla^2\right) \partial_i \vec{B} \right]
$$

$$
= \int d\vec{x} \partial_j \left[a_2 \left(\partial_i \vec{B} \cdot \partial_j \vec{B} - (\partial_j \partial_i \vec{B}) \cdot \vec{B} \right) \right.
$$

$$
+ a_3 \left(\partial_i B_l \nabla^2 \partial_j B_l - \left(\nabla^2 \partial_i \partial_j B_l\right) B_l \right.
$$

$$
\left. \left. + \left(\partial_j \partial_m \partial_i B_l\right) \partial_m B_l - \left(\partial_i \partial_m B_l\right)\left(\partial_j \partial_m B_l\right) \right) + \cdots \right].
$$

$$(52)$$

Therefore,

$$
\frac{dP_i}{dt} = - \int d\vec{x} \left(\alpha \rho \vec{E} + \frac{\vec{J} \times \vec{B}}{c} \right)_i + \frac{1}{4\pi} \oint da \tau_{ij} n_j, \tag{53}
$$

where the stress tensor τ_{ij} is given by

$$
\tau_{ij} = \alpha E_i E_j + B_j f\left(\nabla^2\right) B_i - \frac{\delta_{ij}}{2} \left(\alpha \vec{E} \cdot \vec{E} + \vec{B} f\left(\nabla^2\right) \vec{B} \right)
$$

$$
+ a_2 \left(\partial_i \vec{B} \cdot \partial_j \vec{B} - (\partial_j \partial_i \vec{B}) \cdot \vec{B} \right) + a_3 \left(\partial_i B_l \nabla^2 \partial_j B_l - \left(\nabla^2 \partial_i \partial_j B_l\right) B_l \right.
$$

$$
\left. + \left(\partial_j \partial_m \partial_i B_l\right) \partial_m B_l - \left(\partial_i \partial_m B_l\right)\left(\partial_j \partial_m B_l\right) \right) + \cdots \tag{54}
$$

For $\rho(\vec{x}) = e\delta^3\left(\vec{x} - \vec{x}'\right)$ and $\vec{J}(\vec{x}) = e\vec{v}\delta^3\left(\vec{x} - \vec{x}'\right)$ one obtains

$$
\int d\vec{x} \left(\alpha \rho \vec{E} + \frac{1}{c} \vec{J} \times \vec{B} \right)_i = e\alpha \vec{E} + \frac{e}{c} \vec{v} \times \vec{B}, \tag{55}
$$

which is a modified Lorentz force. Note that when $\vec{B} = 0$ this yields a modified electric force of the form $e'\vec{E}$ with $e' = \alpha e$. That is, the electric charge gets modified to $e' = \alpha e$ and so, for instance, the electric potential of a point charge becomes $\phi = \alpha e/r$. A similar modification occurs to the gravitational potential of a particle [27].

3.3. *The scale transformations*

Note that equations (32)-(33) are invariant under the scale transformations

$$
\vec{E} \to b^{-(z+1)} \vec{E}, \quad \vec{B} \to b^{-2} \vec{B}, \quad t \to b^z t, \quad \vec{x} \to b\vec{x}. \tag{56}
$$

Under these, the action transforms as

$$
S \to b^{(1-z)} S. \tag{57}
$$

Therefore, only if $z = 1$ the transformations (56) are symmetries of the action. In this case, by Noether's theorem, we have the conserved quantity

$$
D_{z=1} = \frac{1}{4\pi c} \int d\vec{x} \left(\alpha \vec{E} \times \vec{B} \right) \cdot \vec{x} - \frac{t}{c} \mathcal{E}, \tag{58}
$$

which is the generator of dilations. For arbitrary z the transformations (56) are not symmetries of the action, but it can be shown that the quantity

$$D = \frac{1}{4\pi c} \int d\vec{x} \left(\alpha \vec{E} \times \vec{B} \right) \cdot \vec{x} - \frac{t}{c}\mathcal{E} + \int dtU, \tag{59}$$

where

$$U = \int d\vec{x} \left(a_2 \vec{B} \cdot \nabla^2 \vec{B} + 2a_3 \vec{B} \cdot \left(\nabla^2\right)^2 \vec{B} + \cdots \right), \tag{60}$$

is conserved. Thus this quantity is related to the scale transformations.

4. Duality transformations and the question of the magnetic monopole

The usual Maxwell's equations in vacuum are invariant under the duality transformations

$$\left(\vec{E}, \vec{B} \right) \rightarrow \left(-\vec{B}, \vec{E} \right). \tag{61}$$

Introducing these into the modified Maxwell's equations (28)-(31) one finds the dual equations

$$\vec{\nabla} \cdot \vec{E} = 0, \qquad \vec{\nabla} \times \left(f\left(\nabla^2\right) \vec{E} \right) = -\frac{\alpha}{c}\frac{\partial \vec{B}}{\partial t}, \tag{62}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \qquad \vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}. \tag{63}$$

Therefore, unlike the usual case, the equations are not self-dual. Now one gets a system of modified Maxwell's equations with the Faraday's law modified. It can be observed that these equations imply the wave equations (34) and (35), and so Eqs. (62)-(63) are also consistent with the dispersion relation (36).

Now, notice that in terms of the scalar magnetic potential $\tilde{\phi}$ and the electric vector potential $\tilde{A}_i$ one has

$$B_i = -\left(\partial_{ct}\tilde{A}_i + \partial_i\tilde{\phi} \right), \quad E_i = \epsilon_{ijk}\partial_j\tilde{A}_k. \tag{64}$$

Dual equations (62)-(63) can be obtained from the dual action

$$S = \int cdtd\vec{x}\mathcal{L} = \int cdtd\vec{x} \left(\alpha B_i B_i - E_i f\left(\nabla^2\right) E_i \right)$$

$$= \int cdtd\vec{x} \left(\alpha B_i B_i - \frac{1}{2}\tilde{F}_{ij}f\left(\nabla^2\right)\tilde{F}_{ij} \right), \tag{65}$$

which is compatible with the dispersion relation (36) and in particular with (2).

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Let us now consider the modified Maxwell's equations including magnetic monopoles

$$\vec{\nabla} \cdot \vec{E} = 4\pi \rho_e, \qquad \vec{\nabla} \times \vec{E} = -\left(\frac{4\pi}{c} \vec{J}_m + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} \right), \tag{66}$$

$$\vec{\nabla} \cdot \vec{B} = 4\pi \rho_m, \qquad \vec{\nabla} \times \left(f\left(\nabla^2\right) \vec{B} \right) = \frac{4\pi}{c} \vec{J}_e + \frac{\alpha}{c} \frac{\partial \vec{E}}{\partial t}. \tag{67}$$

By performing the duality transformations

$$\left(\vec{E}, \vec{B}\right) \to \left(-\vec{B}, \vec{E}\right), \quad \left(\rho_e, \rho_m\right) \to \left(-\rho_m, \rho_e\right), \quad \left(\vec{J}_e, \vec{J}_m\right) \to \left(-\vec{J}_m, \vec{J}_e\right), \tag{68}$$

one obtains the dual equations

$$\vec{\nabla} \cdot \vec{E} = 4\pi \rho_e, \qquad \vec{\nabla} \times \left(f\left(\nabla^2\right) \vec{E} \right) = -\left(\frac{4\pi}{c} \vec{J}_m + \frac{\alpha}{c} \frac{\partial \vec{B}}{\partial t} \right), \tag{69}$$

$$\vec{\nabla} \cdot \vec{B} = 4\pi \rho_m, \qquad \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J}_e + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}. \tag{70}$$

Then, if we have solutions of the equations (66)-(67), by duality transformations (68), solutions to (69)-(70) are obtained.

Some topics on the magnetic monopole for the case of $z = 2$ have been discussed in [50].

5. Green Function

Introducing the definitions of the potentials into (38) and (41) we find

$$\nabla^2 \phi + \frac{1}{c} \frac{\partial \vec{\nabla} \cdot \vec{A}}{\partial t} = -4\pi \rho, \tag{71}$$

$$f\left(\nabla^2\right) \nabla^2 \vec{A} - \frac{\alpha}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla} \cdot \left(f\left(\nabla^2\right) \vec{\nabla} \cdot \vec{A} + \frac{\alpha}{c} \frac{\partial \phi}{\partial t} \right) = -\frac{4\pi}{c} \vec{J}. \tag{72}$$

Using the Coulomb gauge, $\vec{\nabla} \cdot \vec{A} = 0$, in the static case we have

$$\nabla^2 \phi = -4\pi \rho, \quad f\left(\nabla^2\right) \nabla^2 \vec{A} = -\frac{4\pi}{c} \vec{J}. \tag{73}$$

If $f\left(\nabla^2\right) = a_z \left(\nabla^2\right)^{z-1}$ we obtain

$$\nabla^2 \phi = -4\pi \rho, \qquad \left(\nabla^2\right)^z \vec{A} = -\frac{4\pi}{ca_z} \vec{J}. \tag{74}$$

The solution for $\vec{A}$ is given by

$$\vec{A}(\vec{x}) = -\frac{4\pi}{ca_z} \int d\vec{x}_1 G\left(\vec{x}, \vec{x}_1\right) \vec{J}\left(\vec{x}_1\right), \tag{75}$$

where

$$G\left(\vec{x}, \vec{x}_1\right) = \int d\vec{x}_z d\vec{x}_{z-1} \cdots d\vec{x}_2 G_0\left(\vec{x}, \vec{x}_z\right) G_0\left(\vec{x}_z, \vec{x}_{z-1}\right) \cdots G_0\left(\vec{x}_2, \vec{x}_1\right), \tag{76}$$

and

$$G_0 \left(\vec{x}_a, \vec{x}_b \right) = \frac{-1}{4\pi |\vec{x}_a - \vec{x}_b|}, \tag{77}$$

is the Green function for the Maxwell static theory. Remarkably the Green function (76) seems like a quantum correction of order z to the Maxwell's static two-point function [51,52].

As it may be seen, the Lorentz gauge condition is substituted by

$$f \left(\nabla^2 \right) \vec{\nabla} \cdot \vec{A} + \frac{\alpha}{c} \frac{\partial \phi}{\partial t} = 0. \tag{78}$$

In terms of this, one obtains

$$\left(f \left(\nabla^2 \right) \nabla^2 - \frac{\alpha}{c^2} \frac{\partial^2}{\partial t^2} \right) \phi = -4\pi f \left(\nabla^2 \right) \rho, \tag{79}$$

$$\left(f \left(\nabla^2 \right) \nabla^2 - \frac{\alpha}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{A} = -\frac{4\pi}{c} \vec{J}. \tag{80}$$

For this case the potentials are

$$\phi(\vec{x}, t) = \int d\vec{x}' dt' \left[f \left(\vec{\nabla}'^2 \right) \rho(\vec{x}', t') \right] g \left(\vec{R}, \tau \right),$$

$$\vec{A}(\vec{x}, t) = \int d\vec{x}' dt' \vec{J}(\vec{x}') g \left(\vec{R}, \tau \right), \quad \vec{R} = \vec{x} - \vec{x}', \tau = t - t'$$

where

$$g \left(\vec{R}, \tau \right) = -\frac{1}{4\pi^3} \int d\vec{k} d\omega e^{i \left(\vec{k} \cdot \vec{R} - \omega \tau \right)} \tilde{g} \left(k^2, \omega \right) \tag{81}$$

with

$$\tilde{g} \left(k^2, \omega \right) = \frac{c'^2}{\omega^2 - k^2 c'^2 f \left(-k^2 \right)} \tag{82}$$

If $k^2 c'^2 f \left(-k^2 \right) = k^2 c'^2 + a_z \left(k^2 \right)^z$, we can find

$$\tilde{g} \left(k^2, \omega \right) = \frac{c'^2}{\omega^2 - k^2 c'^2 - a_z \left(k^2 \right)^z}. \tag{83}$$

At low energies, the propagator is dominated by the Maxwell theory

$$\tilde{g} \left(k^2, \omega \right) = c'^2 \left(\frac{1}{\omega^2 - k^2 c'^2} + \frac{1}{\omega^2 - k^2 c'^2} a_z \left(k^2 \right)^z \frac{1}{\omega^2 - k^2 c'^2} + \cdots \right). \tag{84}$$

Therefore we can see this theory as a quantum correction to the Maxwell classical theory.

In the high energies regime, the propagator is dominated by $\tilde{g}\left(k^2, \omega\right) = c'^2 / (\omega^2 - a_z \left(k^2\right)^z)$, that is

$$\tilde{g}\left(k^2, \omega\right) = c'^2 \left(\frac{1}{\omega^2 - a_z \left(k^2\right)^z} + \frac{1}{\omega^2 - a_z \left(k^2\right)^z} k^2 c'^2 \frac{1}{\omega^2 - a_z \left(k^2\right)^z} + \cdots \right). \quad (85)$$

The same occurs in the Hořava gravity [9].

6. Summary

In this work we studied an electrodynamics consistent with anisotropic transformations of the space-time with an arbitrary dynamic exponent z. The equations of motion and conserved quantities were obtained. It was shown that the propagator of this theory can be regarded as a quantum correction to the usual Maxwell's propagator. Also, it was shown that the momentum and angular momentum remain unchanged, but their conservation laws have modifications. It was shown that both the speed of light and electric charge run with z. The existence of an additional conserved quantity that changes with z was also shown which, if $z = 1$, is reduced the generator of dilations. In addition, the question of the magnetic monopole was considered, and this lead to showing a dual electrodynamics invariant under the same anisotropic transformations.

In a further work we will study the quantum version of this model. From (85) a better UV behavior than the usual case is expected. Moreover from propagator (83) and equations (84)-(85) we can expect that this model has two fixed points. We hope to prove this affirmation with the renormalization group and study the crossover between these fixed points. We will also couple this field with matter.

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