Schrödinger group and quantum finance

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Abstract

Using the one dimensional free particle symmetries, the quantum finance symmetries are obtained. Namely, it is shown that Black-Scholes equation is invariant under Schrödinger group. In order to do this, the one dimensional free non-relativistic particle and its symmetries are revisited. To get the Black-Scholes equation symmetries, the particle mass is identified as the inverse of square of the volatility. Furthermore, using financial variables, a Schrödinger algebra representation is constructed.

1 Introduction

Lately, mathematical techniques developed in physics have been employed to study systems from other areas. For example, the Black-Scholes [1] and Merton [2] equation is very important to study finance theory and it can be

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mapped to the Schrödinger equation [3]. Then technics that arise in quantum mechanics can be used to study financial phenomena, this fact allowed the birth of a new discipline, the so call Quantum Finance [3]. It is well known that symmetry groups are useful to study different systems. Given that the conformal group is the largest symmetry group of special relativity [4], this group is very important in physics. Furthermore, the Schrödinger group is a non-relativistic conformal group and it is the symmetry group for the free Schrödinger equation [5, 6]. It is worth mentioning that in 1882 Sophus Lie showed that the Fick equation, which describes diffusion, is invariant under the Schrödinger group [7].

In this paper, it will be shown that Black-Scholes equation is invariant under Schrödinger group. In order to do this, the one dimensional free non-relativistic particle and its symmetries will be revisited. The quantum version of the free non-relativistic particle and its symmetries will be revisited too. To get the Black-Scholes equation symmetries, the particle mass is identified as the inverse of square of the volatility. Furthermore, using financial variables, a Schrödinger algebra representation is constructed.

This paper is organized as follows: in section 2 a brief review about one dimensional non-relativistic free particle and its symmetries is given; in section 3 the one dimensional free Schrödinger equation and its symmetries are studied; in section 4 the Black-Scholes equation and its symmetries are studied. Finally, in section 5 a summary is given.

2 Free particle

The one dimensional non-relativistic free particle action is given by

$$S = \int dt \frac{m}{2}\left(\frac{dx}{dt}\right)^2, \tag{1}$$

this is the simplest mechanics system. Now, if $\alpha, \beta, \gamma, \delta, a, v, c$ are constants, the conformal coordinate transformations

$$t' = \frac{\alpha t + \beta}{\gamma t + \delta}, \qquad x' = \frac{ax + vt + c}{\gamma t + \delta}, \qquad a^2 = \alpha\delta - \beta\gamma \neq 0, \tag{2}$$

can be constructed. This coordinates transformation includes temporal translations

$$t' = t + \beta, \qquad x' = x, \tag{3}$$

spatial translations

$$t', \qquad x' = x + c, \tag{4}$$

Galileo's transformations

$$t', \qquad x' = x + vt, \tag{5}$$

anisotropic scaling

$$t' = a^2 t, \qquad x' = ax \tag{6}$$

and the special conformal transformations

$$t' = \frac{1}{\gamma t + 1}, \qquad x' = \frac{x}{\gamma t + 1}. \tag{7}$$

Now, with the coordinates transformation (2), the action

$$S' = \int dt' \frac{m}{2} \left(\frac{dx'}{dt'} \right)^2 \tag{8}$$

can be defined, which satisfies

$$S' = S + \frac{m}{2} \int dt \left(\frac{d\phi(x,t)}{dt} \right), \tag{9}$$

where

$$\phi(x,t) = \frac{1}{a^2} \left(2avx + v^2 t - \frac{\gamma \left(ax + vt + c\right)^2}{\gamma t + \delta} \right). \tag{10}$$

Then, the one dimensional non-relativistic free particle action (1) is invariant under the coordinates transformation (2).

It is shown below that the conformal coordinate transformations (2) and the quantity (10) are used to study Black-Schole equation symmetries, which is important in financial theory.

2.1 Conservative quantities

For the one dimensional non-relativistic particle, the following quantities

$$
\begin{aligned}
P &= m\dot{x}, && (11)\\
H &= \frac{P^2}{2m}, && (12)\\
G &= tP - mx, && (13)\\
K_1 &= tH - \frac{1}{2}xP, && (14)\\
K_2 &= t^2H - txP + \frac{m}{2}x^2 && (15)
\end{aligned}
$$

are conserved.

The momentum P is associated with spatial translations (4). The Hamiltonian H is associated with temporal translations (3) . The quantity G is associated with Galileo's transformations (5). While K_1 is associated with anisotropic scaling (6)and K_2 is associated with the special conformal transformations (7).

Furthermore, using the Poisson brackets, it can be shown that the following relations

$$
\begin{aligned}
\{P, H\} &= 0, && (16)\\
\{P, K_1\} &= \frac{1}{2}P, && (17)\\
\{P, K_2\} &= G, && (18)\\
\{P, G\} &= m, && (19)\\
\{H, K_1\} &= H, && (20)\\
\{H, G\} &= P, && (21)\\
\{H, K_2\} &= 2K_1, && (22)\\
\{K_1, K_2\} &= K_2, && (23)\\
\{K_1, G\} &= \frac{1}{2}G, && (24)\\
\{K_2, G\} &= 0 && (25)
\end{aligned}
$$

are satisfied.

2.2 Schrödinger group

The Schrödinger equation for the one dimensional non-relativistic free particle is

$$i\hbar\frac{\partial\psi\left(x,t\right)}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2}. \tag{26}$$

Now, if the particle is observed in a system with coordinates (x',t'), the particle has to be described by wave function $\psi'(x',t')$, which satisfies

$$i\hbar\frac{\partial\psi'\left(x',t'\right)}{\partial t'}=-\frac{\hbar^2}{2m}\frac{\partial^2\psi'(x',t')}{\partial x'^2}. \tag{27}$$

With a long but straightforward calculation, it can be proved that the Schrödinger equation (26) is invariant under conformal coordinate transformations (2), where the wave function transforms as

$$\psi'\left(x',t'\right)=\left(\sqrt{\gamma t+\delta}\right)e^{\frac{im}{2\hbar}\phi(x,t)}\psi(x,t), \tag{28}$$

here $\phi(x,t)$ is given by (10).

The conformal symmetry for free Schrödinger equation was found by Niederer and Hagen in 1972 [6, 5]. However, this symmetry was obtained by S. Lie in 1882 while he was studying the Fick equation. [7].

Furthermore, according to quantum mechanics, the quantities (11)-(15) are represented by operators

$$\begin{aligned}
\hat{P} &= -i\hbar\frac{\partial}{\partial x}, && (29)\\
\hat{H} &= \frac{\hat{P}^2}{2m}, && (30)\\
\hat{G} &= t\hat{P}-mx, && (31)\\
\hat{K}_1 &= t\hat{H}-\frac{1}{4}\left(x\hat{P}+\hat{P}x\right), && (32)\\
\hat{K}_2 &= t^2\hat{H}-\frac{t}{2}\left(x\hat{P}+\hat{P}x\right)+\frac{m}{2}x^2. && (33)
\end{aligned}$$

Now, whether $[A,B]=AB-BA$, the following algebra

$$\left[\hat{P},\hat{H}\right] = 0, \tag{34}$$

$$\left[\hat{P},\hat{K}_1\right] = \frac{i\hbar}{2}\hat{P}, \tag{35}$$

$$\left[\hat{P},\hat{K}_2\right] = i\hbar\hat{G}, \tag{36}$$

$$\left[\hat{P},\hat{G}\right] = i\hbar m, \tag{37}$$

$$\left[\hat{H},\hat{K}_1\right] = i\hbar\hat{H}, \tag{38}$$

$$\left[\hat{H},\hat{G}\right] = i\hbar\hat{P}, \tag{39}$$

$$\left[\hat{H},\hat{K}_2\right] = 2i\hbar\hat{K}_1, \tag{40}$$

$$\left[\hat{K}_1,\hat{K}_2\right] = i\hbar\hat{K}_2, \tag{41}$$

$$\left[\hat{K}_1,\hat{G}\right] = \frac{i\hbar}{2}\hat{G}, \tag{42}$$

$$\left[\hat{K}_2,\hat{G}\right] = 0 \tag{43}$$

is satisfied, which is similar to the algebra (16)-(25). The algebra (34)-(43) is the so called Schrödinger algebra of Schrödinger group. Now, if $\hat{O}$ is an operator its evolution is given by

$$\frac{d\hat{O}}{dt} = \frac{\partial\hat{O}}{\partial t} + \frac{i}{\hbar}\left[\hat{H},\hat{O}\right]. \tag{44}$$

Using this last equation and the algebra (34)-(43), it is possible to show that operators (29)-(33) are conserved.

In the next section, it will be shown that Schrödinger symmetry arises in Black-Scholes equation too.

3 The Black-Scholes equation

The Black-Scholes equation is [1, 2]

$$\frac{\partial C(s,t)}{\partial t} = -\frac{\sigma^2}{2}s^2\frac{\partial^2 C(s,t)}{\partial s^2} - rs\frac{\partial C(s,t)}{\partial s} + rC(s,t), \tag{45}$$

where C is the price of a derivative, s is the price of the stock, σ is the volatility and r is the annualized risk-free interest rate. This equation is a remarkable result in finance theory.

Amazingly, the Black-Scholes equation (45) is equivalent to Schrödinger equation [3]. In fact, using the change of variable

$$s = e^{x} \tag{46}$$

in equation (45), the following result

$$\frac{\partial C(x,t)}{\partial t} = -\frac{\sigma^2}{2}\frac{\partial^2 C(x,t)}{\partial x^2} + \left(\frac{\sigma^2}{2} - r\right)\frac{\partial C(x,t)}{\partial x} + rC(x,t). \tag{47}$$

is gotten. Additionally, if

$$C(x,t) = e^{\left[\frac{1}{\sigma^2}\left(\frac{\sigma^2}{2}-r\right)x + \frac{1}{2\sigma^2}\left(\frac{\sigma^2}{2}+r\right)^2 t\right]}\psi(x,t) \tag{48}$$

the following equation

$$\frac{\partial \psi(x,t)}{\partial t} = -\frac{\sigma^2}{2}\frac{\partial^2 \psi(x,t)}{\partial x^2} \tag{49}$$

is obtained, wich is like Schrödinger equation (26). Then, since the Schrödinger equation (26) is invariant under conformal transformation (2), the equation (49) is invariant under the same transformations. In this case the function $\psi(x,t)$ transforms as

$$\psi'\left(x',t'\right) = \left(\sqrt{\gamma t + \delta}\right) e^{\frac{1}{2\sigma^2}\phi(x,t)}\psi(x,t), \tag{50}$$

where $\phi(x,t)$ is given by (10). Notice that the particle mass m is changed for $1/\sigma^2$.

3.1 Schrödinger group and Black-Scholes equation

Using the change of variable (46), the coordinates transformations can be written (2) as

$$t' = \frac{\alpha t + \beta}{\gamma t + \delta}, \qquad s' = e^{\left(\frac{bt+c}{\gamma t+\delta}\right)} s^{\left(\frac{a}{\gamma t+\delta}\right)}. \tag{51}$$

Through a long but straightforward calculation, it can be demonstrated that Black-Scholes equation (45) is invariant under this last transformations, where the price $C(s,t)$ transforms as

$$C'\left(s',t'\right) = \left(\sqrt{\gamma t + \delta}\right) s^{\Phi_1(s,t)} e^{\Phi_2(s,t)} C(s,t), \tag{52}$$

here

$$\Phi_1(s,t) = \frac{-2a^2\gamma\left(\frac{\sigma^2}{2}-r\right)t + 2a\left(b\delta-\gamma c\right) + 2a^2(a-\delta)\left(\frac{\sigma^2}{2}-r\right)}{2a^2\sigma^2\left(\gamma t+\delta\right)} - \frac{\gamma a^2\left(\ln s\right)}{2a^2\sigma^2\left(\gamma t+\delta\right)}$$

and

$$\Phi_2(s,t) = \left(\frac{a^2\left(\frac{\sigma^2}{2}+r\right)^2(\alpha-\delta) + 2a^2b\left(\frac{\sigma^2}{2}-r\right) + b\left(b\delta-2\gamma c\right)}{2\sigma^2a^2(\gamma t+\delta)}\right)t + \frac{a^2\beta\left(\frac{\sigma^2}{2}+r\right)^2 + 2a^2\left(\frac{\sigma^2}{2}-r\right)c - \gamma c^2 - \gamma a^2\left(\frac{\sigma^2}{2}+r\right)^2 t^2}{2\sigma^2a^2(\gamma t+\delta)}.$$

Now, the Black-Scholes equation (45) can be written as

$$\frac{\partial C(s,t)}{\partial t} = \hat{\mathbf{H}}C(s,t). \tag{53}$$

where

$$\hat{\mathbf{H}} = -\frac{\sigma^2}{2}s^2\frac{\partial^2}{\partial s^2} - rs\frac{\partial}{\partial s} + r. \tag{54}$$

Moreover, using the operator

$$\hat{\Pi} = -is\frac{\partial}{\partial s} + \frac{i}{\sigma^2}\left(\frac{\sigma^2}{2}-r\right), \tag{55}$$

the operator $\hat{\mathbf{H}}$ can be rewritten as

$$\hat{\mathbf{H}} = \frac{\sigma^2}{2}\hat{\Pi}^2 + \frac{1}{2\sigma^2}\left(\frac{\sigma^2}{2}+r\right)^2. \tag{56}$$

Notice that operator $\hat{\mathbf{H}}$ is similar to Hamiltonian operator $\hat{H}$ (30), where the particle mass m is associated with $1/\sigma^2$.

Additionally, using the operator (55) it is possible construct quantities related with the non-relativistic free particle conserved quantities (29)-(33). In fact, operators

$$\begin{aligned}
\hat{\Pi} &= -is\frac{\partial}{\partial s} + \frac{i}{\sigma^2}\left(\frac{\sigma^2}{2} - r\right), && (57)\\
\hat{\mathbf{H}}_0 &= \frac{\sigma^2}{2}\hat{\Pi}^2, && (58)\\
\hat{\mathbf{G}} &= t\hat{\Pi} - \frac{1}{\sigma^2}\ln s, && (59)\\
\hat{\mathbf{K}}_1 &= t\hat{\mathbf{H}}_0 - \frac{1}{4}\left(\ln s\hat{\Pi} + \hat{\Pi}\ln s\right), && (60)\\
\hat{\mathbf{K}}_2 &= t^2\hat{\mathbf{H}}_0 - \frac{t}{2}\left(\ln s\hat{\Pi} + \hat{\Pi}\ln s\right) + \frac{1}{2\sigma^2}(\ln s)^2 && (61)
\end{aligned}$$

can be proposed, which are similar to quantities (29)-(33). Furthermore, using the relation

$$[\ln s, \hat{\Pi}] = i, \qquad (62)$$

the algebra

$$\begin{aligned}
\left[\hat{\Pi}, \hat{\mathbf{H}}\right] &= 0, && (63)\\
\left[\hat{\Pi}, \hat{\mathbf{K}}_1\right] &= \frac{i}{2}\hat{\Pi}, && (64)\\
\left[\hat{\Pi}, \hat{\mathbf{K}}_2\right] &= i\hat{\mathbf{G}}, && (65)\\
\left[\hat{\Pi}, \hat{\mathbf{G}}\right] &= \frac{i}{\sigma^2}, && (66)\\
\left[\hat{\mathbf{H}}, \hat{\mathbf{K}}_1\right] &= i\hat{\mathbf{H}}_0 && (67)\\
\left[\hat{\mathbf{H}}, \hat{\mathbf{G}}\right] &= i\hat{\Pi}, && (68)\\
\left[\hat{\mathbf{H}}, \hat{\mathbf{K}}_2\right] &= 2i\hat{\mathbf{K}}_1, && (69)\\
\left[\hat{\mathbf{K}}_1, \hat{\mathbf{K}}_2\right] &= i\hat{\mathbf{K}}_2, && (70)\\
\left[\hat{\mathbf{K}}_1, \hat{\mathbf{G}}\right] &= \frac{i}{2}\hat{\mathbf{G}}, && (71)\\
\left[\hat{\mathbf{K}}_2, \hat{\mathbf{G}}\right] &= 0. && (72)
\end{aligned}$$

is satisfied. Then the operators (57)-(61) satisfy the Schrödinger algebra.

Another study about Black-Scholes symmetries can be seen in [8].

4 Summary

It was shown that Black-Scholes equation is invariant under Schrödinger group. In order to do this, the one dimensional free non-relativistic particle and its symmetries were revisited. The quantum version of the free non-relativistic particle and its symmetries were revisited too. To get the Black-Scholes equation symmetries, the particle mass was identified as the inverse of square of the volatility. Besides, using financial variables, a Schrödinger algebra representation was constructed. This result shows that physical techniques can be employed to study other disciplines.

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