# The Black-Scholes Equation and Certain Quantum Hamiltonians 

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#### Abstract

In this paper a quantum mechanics is built by means of a nonHermitian momentum operator. We have shown that it is possible to construct two Hermitian and two non-Hermitian type of Hamiltonians using this momentum operator. We can construct a generalized supersymmetric quantum mechanics that has a dual based on these Hamiltonians. In addition, it is shown that the non-Hermitian Hamiltonians of this theory can be related to Hamiltonians that naturally arise in the so-called quantum finance.


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## 1 Introduction

In the origins of quantum mechanics P.A.M Dirac [6] observed that commutation relations

$$
\begin{equation*}
\left[x_{i}, x_{j}\right]=\left[P_{i}, P_{j}\right]=0, \quad\left[x_{i}, P_{j}\right]=i \delta_{i j}, \tag{1}
\end{equation*}
$$

are satisfied by the operators $x_{i}, P_{j}=-i \partial_{j}$ and also by the set

$$
\begin{equation*}
x_{i}, \quad P_{(f) j}=-i \partial_{j}+i \partial_{j} f \tag{2}
\end{equation*}
$$

with $f$ been an arbitrary function [6]. For different reasons, operators (2) were discarded, for example, the Hamiltonian $H_{f}=\frac{P_{f}^{2}}{2}$ is non-Hermitian and it could have a non-real spectrum. However, it has recently been shown that there are non-Hermitian operators with real spectrum [3]. Studies of non-Hermitian Hamiltonians and their applications in physics can be found in the papers [7], [4] and [10].

In this paper, we will show that, using the operator $P_{(f) j}$, four different types of Hamiltonians can be built, two of these Hermitians and the other two, non-Hermitians. We will show that, from two Hermitian Hamiltonians in one dimension, it is possible to construct a supersymmetric mechanics, and that using one of the two non-Hermitian Hamiltonians a generalized supersymmetric mechanic can be constructed. Moreover, we will show that this new supersymmetric quantum mechanics has a dual and the ground state of the corresponding Hamiltonians will be found.

As a second point of this work, it is shown how the operator $P_{(f) j}$ can also be used to build some of non-Hermitian Hamiltonians that naturally arise in the so-called quantum finance.

## 2 Non-Hermitian Hamiltonians

In this section, we will study the quantum mechanics that emerges when the operator $P_{(f) j}$ is considered. As an starting point, we have to notice that the operator $P_{(f) i}$ is given by the transformation

$$
\begin{equation*}
P_{(f) i}=e^{f} P_{i} e^{-f} \tag{3}
\end{equation*}
$$

and also that it is not Hermitian.
With $\vec{P}_{(f)}$ we can construct four Hamiltonians, two of them Hermitians

$$
\begin{align*}
& H_{1}=\alpha^{2} \vec{P}_{(f)}^{\dagger} \cdot \vec{P}_{(f)}=\alpha\left(\vec{P}^{2}+\nabla^{2} f+(\vec{\nabla} f)^{2}\right)  \tag{4}\\
& H_{2}=\alpha^{2} \vec{P}_{(f)} \cdot \vec{P}_{(f)}^{\dagger}=\alpha\left(\vec{P}^{2}-\nabla^{2} f+(\vec{\nabla} f)^{2}\right) \tag{5}
\end{align*}
$$

and two non-Hermitians

$$
\begin{align*}
H_{3} & =\beta^{2} \vec{P}_{(f)}^{\dagger} \cdot \vec{P}_{(f)}^{\dagger} \\
& =\beta^{2}\left(\vec{P}^{2}-2 i \vec{\nabla} f \cdot \vec{P}-\nabla^{2} f-(\vec{\nabla} f)^{2}\right),  \tag{6}\\
H_{4} & =\beta^{2} \vec{P}_{(f)} \cdot \vec{P}_{(f)} \\
& =\beta^{2}\left(\vec{P}^{2}+2 i \vec{\nabla} f \cdot \vec{P}+\nabla^{2} f-(\vec{\nabla} f)^{2}\right) . \tag{7}
\end{align*}
$$

These Hamiltonians are obtained naturally in different contexts. In the following subsection, we will see that, they can be used to obtain a generalized version of the supersymmetric quantum mechanics.

## 3 Supersymmetric Quantum Mechanics

In the one dimensional case, the Hamiltonians $H_{1}$ and $H_{2}$ are given by

$$
\begin{align*}
& H_{1}=\alpha^{2}\left(P^{2}+\frac{d^{2} f}{d x^{2}}+\left(\frac{d f}{d x}\right)^{2}\right)  \tag{8}\\
& H_{2}=\alpha^{2}\left(P^{2}-\frac{d^{2} f}{d x^{2}}+\left(\frac{d f}{d x}\right)^{2}\right) \tag{9}
\end{align*}
$$

Moreover, if

$$
\begin{equation*}
f(x)=\int_{0}^{x} W(u) d u \tag{10}
\end{equation*}
$$

then

$$
\begin{equation*}
H_{1}=\alpha^{2}\left(P^{2}+\frac{d W}{d x}+W^{2}\right) \tag{11}
\end{equation*}
$$

$$
\begin{equation*}
H_{2}=\alpha^{2}\left(P^{2}-\frac{d W}{d x}+W^{2}\right) \tag{12}
\end{equation*}
$$

This Hamiltonians can be used to form the matrix

$$
h=\left(\begin{array}{rr}
H_{1} & 0  \tag{14}\\
0 & H_{2}
\end{array}\right)
$$

Now, defining

$$
Q=\left(\begin{array}{rr}
0 & \alpha P_{(f)}  \tag{15}\\
0 & 0
\end{array}\right)
$$

we have

$$
\begin{equation*}
h=\left\{Q, Q^{\dagger}\right\}, \quad Q^{2}=0, \quad\{Q, H\}=0 \tag{16}
\end{equation*}
$$

According to the supersymmetric quantum mechanics [5], $h$ represents a superhamiltonian and $Q$ a supercharge. Therefore, the quantum mechanics built using $P_{(f) i}$ contains the usual supersymmetric quantum mechanics.

Moreover, $P_{(f)}$ allows us to generalize supersymmetric quantum mechanics. In fact, we can define the matrices

$$
Q_{1}=\left(\begin{array}{rrrr}
0 & \alpha P_{(f)} & 0 & 0  \tag{17}\\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \beta P_{(f)}^{\dagger} \\
0 & 0 & 0 & 0
\end{array}\right), Q_{2}=\left(\begin{array}{rrrr}
0 & 0 & 0 & 0 \\
\alpha P_{(f)}^{\dagger} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \beta P_{(f)}^{\dagger} & 0
\end{array}\right)
$$

and then $Q_{1}^{2}=Q_{2}^{2}=0$. Using $Q_{1}^{2}=Q_{2}^{2}=0$ we can construct the Hamiltonian

$$
H=\left\{Q_{1}, Q_{2}\right\}=\left(\begin{array}{rrrr}
H_{1} & 0 & 0 & 0  \tag{18}\\
0 & H_{2} & 0 & 0 \\
0 & 0 & H_{3} & 0 \\
0 & 0 & 0 & H_{3}
\end{array}\right)
$$

with the conserved charges

$$
\begin{equation*}
\dot{Q}_{1}=\left[Q_{1}, H\right]=0, \quad \dot{Q}_{2}=\left[Q_{2}, H\right]=0 \tag{19}
\end{equation*}
$$

now, if $\beta=0$, we have $Q_{1}=Q_{2}=Q$ and this quantum mechanics reduces to the usual supersymmetric quantum mechanics.

Besides, we have another quantum mechanics that reduces to the usual supersymmetric quantum mechanics. In fact, if

$$
Q_{3}=\left(\begin{array}{rrrr}
0 & \alpha P_{(f)}^{\dagger} & 0 & 0  \tag{20}\\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \beta P_{(f)} \\
0 & 0 & 0 & 0
\end{array}\right), Q_{4}=\left(\begin{array}{rrrr}
0 & 0 & 0 & 0 \\
\alpha P_{(f)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \beta P_{(f)} & 0
\end{array}\right)
$$

then $Q_{3}^{2}=Q_{4}^{2}=0$ and we can construct the Hamiltonian

$$
\tilde{H}=\left\{Q_{3}, Q_{4}\right\}=\left(\begin{array}{rrrr}
H_{2} & 0 & 0 & 0  \tag{21}\\
0 & H_{1} & 0 & 0 \\
0 & 0 & H_{4} & 0 \\
0 & 0 & 0 & H_{4}
\end{array}\right)
$$

Note that if $\beta=0$ this Hamiltonian is just the usual superhamiltonian $h$.
If we make the transformation $f \rightarrow-f$, we have

$$
\begin{equation*}
\left(H_{1}, H_{2}, H_{3}, H_{4}\right) \quad \rightarrow \quad\left(H_{2}, H_{1}, H_{4}, H_{3}\right) \tag{22}
\end{equation*}
$$

i.e

$$
\begin{equation*}
H \rightarrow \tilde{H} \tag{23}
\end{equation*}
$$

Then, there is a duality transformation between Hamiltonians $H$ and $\tilde{H}$. Therefore these generalized quantum mechanics are duals.

Now, if we consider the functions $\psi_{0}=A_{1} e^{f}, \phi_{0}=A_{2} e^{-f}$ that satisfy

$$
\begin{equation*}
P_{(f)} \psi_{0}=0, \quad P_{(f)}^{\dagger} \phi_{0}=0 \tag{24}
\end{equation*}
$$

Then, the wave function

$$
\psi=\left(\begin{array}{c}
A_{1} e^{-f}  \tag{25}\\
A_{2} e^{f} \\
A_{3} e^{-f} \\
A_{3} e^{-f}
\end{array}\right)
$$

satisfies

$$
\begin{equation*}
H \psi=0 . \tag{26}
\end{equation*}
$$

Moreover, the wave function

$$
\tilde{\psi}=\left(\begin{array}{c}
e^{f}  \tag{27}\\
e^{-f} \\
e^{f} \\
e^{f}
\end{array}\right)
$$

satisfies

$$
\begin{equation*}
\tilde{H} \tilde{\psi}=0 . \tag{28}
\end{equation*}
$$

Thus, $\psi$ is the ground state of $H$ and $\tilde{\psi}$ is the ground state of $\tilde{H}$.

## 4 The Black-Scholes model

This model is a partial differential equation whose solution describes the value of an European Option. See [2], 8]. Nowadays, it is widely used to estimate the pricing of options other than the European ones. Let $\left(\Omega, \mathcal{F}, P, \mathcal{F}_{t \geq 0}\right)$ be a filtered probability space and let $W_{t}$ be a brownian motion in $\mathbf{R}$. We will consider the stochastic differential equation (s.d.e.)

$$
\begin{equation*}
d X(t)=a(t, X(t)) d t+\sigma(t, X(t)) d W(t) \tag{29}
\end{equation*}
$$

with $a$ and $\sigma$ continuous in $(t, x)$ and Lipschitz in $x$. The price processes given by the geometric brownian motion $S(t), S(0)=x_{0}$, solution of the s.d.e.

$$
\begin{equation*}
d S(t)=\mu S(t) d t+\sigma S(t) d W(t) \tag{30}
\end{equation*}
$$

with $\mu$ and $\sigma$ constants. It is well know the solution of this s.d.e. it is given by:

$$
\begin{equation*}
d S(t)=x_{0} \exp \left\{\sigma\left(W(t)-W\left(t_{0}\right)\right)+\left(r-\frac{1}{2} \sigma^{2}\right)\left(t-t_{0}\right)\right\} \tag{31}
\end{equation*}
$$

Let $0 \leq t<T$ and $h$ be a Borel measurable function, $h(X(T))$ denote the contingent claim, let $E^{x, t} h(X(T))$ be the expectation of $h(X(T))$, with the initial condition $X(t)=x$.

Now we recall the Feynman-Kac theorem [9]. Let $v(t, x)=E^{x, t} h(X(T))$ be, $0 \leq t<T$, where $d X(t)=a(X(t)) d t+\sigma(X(t)) d W(t)$. Then

$$
\begin{equation*}
v_{t}(t, x)+a(x) v_{x}(t, x)+\frac{1}{2} \sigma^{2}(x) v_{x x}(t, x)=0, \text { and } \quad \mathrm{v}(\mathrm{~T}, \mathrm{x})=\mathrm{h}(\mathrm{x}) . \tag{32}
\end{equation*}
$$

Now, if we consider the discounted value

$$
u(t, x)=e^{-r(T-t)} E^{x, t} h(X(T))=e^{-r(T-t)} v(t, x)
$$

Then if at time $t, S(t)=x$, if we proceed in standard way,

$$
\begin{aligned}
v(t, x) & =e^{r(T-t)} u(t, x) \\
v_{t}(t, x) & =-r e^{r(T-t)} u(t, x)+e^{r(T-t)} u_{t}(t, x) \\
v_{x}(t, x) & =e^{r(T-t)} u_{x}(t, x) \\
v_{x x}(t, x) & =e^{r(T-t)} u_{x x}(t, x)
\end{aligned}
$$

The Black-Scholes equation is obtained substituting the above equalities in the equation (32) and multiplying by the factor $e^{-r(T-t)}$ :

$$
\begin{gather*}
-r u(t, x)+u_{t}(t, x)+r x u_{x}(t, x)+\frac{1}{2} \sigma^{2} x^{2} v_{x x}(t, x)=0  \tag{33}\\
0 \leq t<T, x \geq 0
\end{gather*}
$$

## 5 The Relation with the Black-Scholes Equation

The operators $\vec{P}_{f} \cdot \vec{P}_{f}$ and $\vec{P}_{f}^{\dagger} \cdot \vec{P}_{f}^{\dagger}$ are non-Hermitians, using them only nonHermitians Hamiltonians such as $H_{3}$ and $H_{4}$, can be constructed. However, we will show that these operators may have applications in some other areas such as quantum finance. In order to see this, we define the potentials

$$
\begin{align*}
& U_{1}(x, y, z)=-\beta^{2}\left(\nabla^{2} f-(\vec{\nabla} f)^{2}\right)+V_{1}(x, y, z)  \tag{34}\\
& U_{2}(x, y, z)=\beta^{2}\left(\nabla^{2} f+(\vec{\nabla} f)^{2}\right)+V_{2}(x, y, z) \tag{35}
\end{align*}
$$

and the non-Hermitians Hamiltonians

$$
\begin{align*}
H_{I} & =\beta^{2} \vec{P}_{(f)} \cdot \vec{P}_{(f)}+U_{1}(x, y, z) \\
& =\beta^{2}\left(\vec{P}^{2}+2 i \vec{\nabla} f \cdot \vec{P}\right)+V_{1}(x, y, z),  \tag{36}\\
H_{I I} & =\beta^{2} \vec{P}_{(f)}^{\dagger} \cdot \vec{P}_{(f)}^{\dagger}+U_{2}(x, y, z) \\
& =\beta^{2}\left(\vec{P}^{2}-2 i \vec{\nabla} f \cdot \vec{P}\right)+V_{2}(x, y, z) . \tag{37}
\end{align*}
$$

On the other hand, let us consider the fundamental equation in quantum finance, the so-called Black-Scholes equation (33)

$$
\begin{equation*}
\frac{\partial C}{\partial t}=-\frac{\sigma^{2} S^{2}}{2} \frac{\partial^{2} C}{\partial S^{2}}-r S \frac{\partial C}{\partial S}+r C \tag{38}
\end{equation*}
$$

where $C$ is the option price, $\sigma$ is a constant called the volatility and $r$ is the interest rate [1]. With the change of variable $S=e^{x}$ we obtain

$$
\begin{align*}
\frac{\partial C}{\partial t} & =H_{B S} C \\
H_{B S} & =-\frac{\sigma^{2}}{2} \frac{\partial^{2}}{\partial x^{2}}+\left(\frac{\sigma^{2}}{2}-r\right) \frac{\partial}{\partial x}+r \tag{39}
\end{align*}
$$

this non-Hermitian Hamiltonian is called Black-Scholes Hamiltonian. Now, considering the one dimensional case of (37) and identifying

$$
\beta^{2}=\frac{\sigma^{2}}{2}, \quad f(x)=\frac{1}{\sigma^{2}}\left(\frac{\sigma^{2}}{2}-r\right) x, \quad V_{2}(x)=r
$$

we obtain $H_{I I}=H_{B S}$.
One generalized Black-Scholes equation, (see [1]) is given by

$$
\begin{equation*}
H_{B S G}=-\frac{\sigma^{2}}{2} \frac{\partial^{2}}{\partial x^{2}}+\left(\frac{\sigma^{2}}{2}-V(x)\right) \frac{\partial}{\partial x}+V(x) \tag{40}
\end{equation*}
$$

In this case, considering again the one dimensional case of equation (37) and with

$$
\beta^{2}=\frac{\sigma^{2}}{2}, \quad f(x)=\int_{0}^{x} d u \frac{1}{\sigma^{2}}\left(\frac{\sigma^{2}}{2}-V(u)\right), \quad V_{2}(x)=V(x)
$$

we have $H_{I I}=H_{B S G}$.
Moreover, the so-called barrier option case has Hamiltonian

$$
\begin{equation*}
H_{B S B}=-\frac{\sigma^{2}}{2} \frac{\partial^{2}}{\partial x^{2}}+\left(\frac{\sigma^{2}}{2}-r\right) \frac{\partial}{\partial x}+V(x) \tag{41}
\end{equation*}
$$

Again, considering (37) in one dimension and with

$$
\beta^{2}=\frac{\sigma^{2}}{2}, \quad f(x)=\frac{1}{\sigma^{2}}\left(\frac{\sigma^{2}}{2}-r\right) x, \quad V_{2}(x)=V(x)
$$

we have $H_{I I}=H_{B S B}$.
As we have seen, several important Hamiltonians appearing in quantum finance are particular cases of this new version of quantum mechanics.

## 6 Summary

A quantum mechanics is built by means of a non-Hermitian momentum operator. Moreover, It is shown that using this momentum operator it is possible to construct two Hermitian and two non-Hermitian type of Hamiltonians. Using these Hermitian Hamiltonians we have built, a generalized supersymmetric quantum mechanics with a dual that can be constructed. It also shown that, the non-Hermitian Hamiltonians of this theory may be related to so-called quantum finance Hamiltonian.

## References

[1] B. E. Baaquie, Quantum Finance, Cambridge University Press (2004).
[2] F. Black and M. Scholes, The Pricing Options and Corporate Liabilities, Journal of Political Economy (81), 637-659 (1973).
[3] C. M. Bender, Introduction to PT-Symmetric Quantum Theory, Contemp. Phys. 46, 277 (2005).
[4] C. M. Bender, P. D. Mannheim, No-ghost theorem for the fourth-order derivative Pais-Uhlenbeck oscillator model Phys. Rev. Lett. 100, 110402 (2008).
[5] F. Cooper, A. Khare, U. Sukhatme, Supersymmetry and Quantum Mechanics, Phys. Rept. 251, 267-385 (1995).
[6] P.A. M. Dirac, The Principles of Quantum Mechanics, Oxford (1930).
[7] P. K. Ghosh, On the construction of a pseudo-Hermitian quantum system with a pre-determined metric in the Hilbert space, J. Phys. A, Math. Theor. 43, 125203 (2010).
[8] R.C. Merton Theory of Rational Options Pricing, Bell Journal of Economic and Management Science (4), 141-183, (1973).
[9] S. Shreve, Stochastic Calculus for Finance II: Continuous-Time Models (Springer Finance) (2004).
[10] T. Tanaka, General aspects of PT-symmetric and P-self-adjoint quantum theory in a Krein space, J. Phys. A 39, 14175 (2006).


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