The Black-Scholes Equation and Certain Quantum Hamiltonians

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Abstract

In this paper a quantum mechanics is built by means of a non-Hermitian momentum operator. We have shown that it is possible to construct two Hermitian and two non-Hermitian type of Hamiltonians using this momentum operator. We can construct a generalized supersymmetric quantum mechanics that has a dual based on these Hamiltonians. In addition, it is shown that the non-Hermitian Hamiltonians of this theory can be related to Hamiltonians that naturally arise in the so-called quantum finance.

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1 Introduction

In the origins of quantum mechanics P.A.M Dirac [6] observed that commutation relations

$$[x_i, x_j] = [P_i, P_j] = 0, \qquad [x_i, P_j] = i\delta_{ij}, \tag{1}$$

are satisfied by the operators $x_i, P_j = -i\partial_j$ and also by the set

$$x_i, \qquad P_{(f)j} = -i\partial_j + i\partial_j f, \tag{2}$$

with f been an arbitrary function [6]. For different reasons, operators (2) were discarded, for example, the Hamiltonian $H_f = \frac{P_f^2}{2}$ is non-Hermitian and it could have a non-real spectrum. However, it has recently been shown that there are non-Hermitian operators with real spectrum [3]. Studies of non-Hermitian Hamiltonians and their applications in physics can be found in the papers [7], [4] and [10].

In this paper, we will show that, using the operator $P_{(f)j}$, four different types of Hamiltonians can be built, two of these Hermitians and the other two, non-Hermitians. We will show that, from two Hermitian Hamiltonians in one dimension, it is possible to construct a supersymmetric mechanics, and that using one of the two non-Hermitian Hamiltonians a generalized supersymmetric mechanic can be constructed. Moreover, we will show that this new supersymmetric quantum mechanics has a dual and the ground state of the corresponding Hamiltonians will be found.

As a second point of this work, it is shown how the operator $P_{(f)j}$ can also be used to build some of non-Hermitian Hamiltonians that naturally arise in the so-called quantum finance.

2 Non-Hermitian Hamiltonians

In this section, we will study the quantum mechanics that emerges when the operator $P_{(f)j}$ is considered. As an starting point, we have to notice that the operator $P_{(f)i}$ is given by the transformation

$$P_{(f)i} = e^f P_i e^{-f}, (3)$$

and also that it is not Hermitian.

With $\vec{P}_{(f)}$ we can construct four Hamiltonians, two of them Hermitians

$$H_1 = \alpha^2 \vec{P}_{(f)}^{\dagger} \cdot \vec{P}_{(f)} = \alpha \left(\vec{P}^2 + \nabla^2 f + \left(\vec{\nabla} f \right)^2 \right), \tag{4}$$

$$H_2 = \alpha^2 \vec{P}_{(f)} \cdot \vec{P}_{(f)}^{\dagger} = \alpha \left(\vec{P}^2 - \nabla^2 f + \left(\vec{\nabla} f \right)^2 \right)$$
(5)

and two non-Hermitians

$$H_{3} = \beta^{2} \vec{P}_{(f)}^{\dagger} \cdot \vec{P}_{(f)}^{\dagger}$$

$$= \beta^{2} \left(\vec{P}^{2} - 2i \vec{\nabla} f \cdot \vec{P} - \nabla^{2} f - \left(\vec{\nabla} f \right)^{2} \right), \qquad (6)$$

$$H_{4} = \beta^{2} \vec{P}_{(f)} \cdot \vec{P}_{(f)}$$

These Hamiltonians are obtained naturally in different contexts. In the following subsection, we will see that, they can be used to obtain a generalized version of the supersymmetric quantum mechanics.

3 Supersymmetric Quantum Mechanics

In the one dimensional case, the Hamiltonians H_1 and H_2 are given by

$$H_1 = \alpha^2 \left(P^2 + \frac{d^2 f}{dx^2} + \left(\frac{df}{dx}\right)^2 \right), \qquad (8)$$

$$H_2 = \alpha^2 \left(P^2 - \frac{d^2 f}{dx^2} + \left(\frac{df}{dx}\right)^2 \right).$$
(9)

Moreover, if

$$f(x) = \int_0^x W(u)du,$$
(10)

then

$$H_1 = \alpha^2 \left(P^2 + \frac{dW}{dx} + W^2 \right), \qquad (11)$$

$$H_2 = \alpha^2 \left(P^2 - \frac{dW}{dx} + W^2 \right).$$
 (12)

(13)

This Hamiltonians can be used to form the matrix

$$h = \begin{pmatrix} H_1 & 0\\ 0 & H_2 \end{pmatrix}. \tag{14}$$

Now, defining

$$Q = \begin{pmatrix} 0 & \alpha P_{(f)} \\ 0 & 0 \end{pmatrix},\tag{15}$$

we have

$$h = \{Q, Q^{\dagger}\}, \qquad Q^2 = 0, \quad \{Q, H\} = 0.$$
 (16)

According to the supersymmetric quantum mechanics [5], h represents a superhamiltonian and Q a supercharge. Therefore, the quantum mechanics built using $P_{(f)i}$ contains the usual supersymmetric quantum mechanics.

Moreover, $P_{(f)}$ allows us to generalize supersymmetric quantum mechanics. In fact, we can define the matrices

$$Q_{1} = \begin{pmatrix} 0 & \alpha P_{(f)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta P_{(f)}^{\dagger} \\ 0 & 0 & 0 & 0 \end{pmatrix}, Q_{2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \alpha P_{(f)}^{\dagger} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \beta P_{(f)}^{\dagger} & 0 \end{pmatrix}, \quad (17)$$

and then $Q_1^2 = Q_2^2 = 0$. Using $Q_1^2 = Q_2^2 = 0$ we can construct the Hamiltonian

$$H = \{Q_1, Q_2\} = \begin{pmatrix} H_1 & 0 & 0 & 0\\ 0 & H_2 & 0 & 0\\ 0 & 0 & H_3 & 0\\ 0 & 0 & 0 & H_3 \end{pmatrix}$$
(18)

with the conserved charges

$$\dot{Q}_1 = [Q_1, H] = 0, \qquad \dot{Q}_2 = [Q_2, H] = 0;$$
(19)

now, if $\beta = 0$, we have $Q_1 = Q_2 = Q$ and this quantum mechanics reduces to the usual supersymmetric quantum mechanics.

Besides, we have another quantum mechanics that reduces to the usual supersymmetric quantum mechanics. In fact, if

$$Q_{3} = \begin{pmatrix} 0 & \alpha P_{(f)}^{\dagger} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta P_{(f)} \\ 0 & 0 & 0 & 0 \end{pmatrix}, Q_{4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \alpha P_{(f)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \beta P_{(f)} & 0 \end{pmatrix}, \quad (20)$$

then $Q_3^2 = Q_4^2 = 0$ and we can construct the Hamiltonian

$$\tilde{H} = \{Q_3, Q_4\} = \begin{pmatrix} H_2 & 0 & 0 & 0\\ 0 & H_1 & 0 & 0\\ 0 & 0 & H_4 & 0\\ 0 & 0 & 0 & H_4 \end{pmatrix}.$$
(21)

Note that if $\beta = 0$ this Hamiltonian is just the usual superhamiltonian h.

If we make the transformation $f \to -f$, we have

$$(H_1, H_2, H_3, H_4) \to (H_2, H_1, H_4, H_3),$$
 (22)

i.e

$$H \to \tilde{H}.$$
 (23)

Then, there is a duality transformation between Hamiltonians H and H. Therefore these generalized quantum mechanics are duals.

Now, if we consider the functions $\psi_0 = A_1 e^f$, $\phi_0 = A_2 e^{-f}$ that satisfy

$$P_{(f)}\psi_0 = 0, \qquad P_{(f)}^{\dagger}\phi_0 = 0.$$
 (24)

Then, the wave function

$$\psi = \begin{pmatrix} A_1 e^{-f} \\ A_2 e^f \\ A_3 e^{-f} \\ A_3 e^{-f} \end{pmatrix}$$
(25)

satisfies

$$H\psi = 0. \tag{26}$$

Moreover, the wave function

$$\tilde{\psi} = \begin{pmatrix} e^f \\ e^{-f} \\ e^f \\ e^f \end{pmatrix}$$
(27)

satisfies

$$\tilde{H}\tilde{\psi} = 0. \tag{28}$$

Thus, ψ is the ground state of H and $\tilde{\psi}$ is the ground state of \tilde{H} .

4 The Black-Scholes model

This model is a partial differential equation whose solution describes the value of an European Option. See [2], [8]. Nowadays, it is widely used to estimate the pricing of options other than the European ones. Let $(\Omega, \mathcal{F}, P, \mathcal{F}_{t\geq 0})$ be a filtered probability space and let W_t be a brownian motion in **R**. We will consider the stochastic differential equation (s.d.e.)

$$dX(t) = a(t, X(t))dt + \sigma(t, X(t))dW(t),$$
(29)

with a and σ continuous in (t, x) and Lipschitz in x. The price processes given by the geometric brownian motion S(t), $S(0) = x_0$, solution of the s.d.e.

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \qquad (30)$$

with μ and σ constants. It is well know the solution of this s.d.e. it is given by:

$$dS(t) = x_0 \exp\{\sigma(W(t) - W(t_0)) + (r - \frac{1}{2}\sigma^2)(t - t_0)\}$$
(31)

Let $0 \le t < T$ and h be a Borel measurable function, h(X(T)) denote the contingent claim, let $E^{x,t}h(X(T))$ be the expectation of h(X(T)), with the initial condition X(t) = x.

Now we recall the Feynman–Kac theorem [9]. Let $v(t, x) = E^{x,t}h(X(T))$ be, $0 \le t < T$, where $dX(t) = a(X(t))dt + \sigma(X(t))dW(t)$. Then

$$v_t(t,x) + a(x)v_x(t,x) + \frac{1}{2}\sigma^2(x)v_{xx}(t,x) = 0$$
, and $v(T,x) = h(x)$. (32)

Now, if we consider the discounted value

$$u(t,x) = e^{-r(T-t)}E^{x,t}h(X(T)) = e^{-r(T-t)}v(t,x).$$

Then if at time t, S(t) = x, if we proceed in standard way,

$$v(t,x) = e^{r(T-t)}u(t,x),$$

$$v_t(t,x) = -re^{r(T-t)}u(t,x) + e^{r(T-t)}u_t(t,x),$$

$$v_x(t,x) = e^{r(T-t)}u_x(t,x),$$

$$v_{xx}(t,x) = e^{r(T-t)}u_{xx}(t,x).$$

The Black-Scholes equation is obtained substituting the above equalities in the equation (32) and multiplying by the factor $e^{-r(T-t)}$:

$$-ru(t,x) + u_t(t,x) + rxu_x(t,x) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x) = 0,$$
(33)
$$0 \le t < T, x \ge 0.$$

5 The Relation with the Black-Scholes Equation

The operators $\vec{P}_f \cdot \vec{P}_f$ and $\vec{P}_f^{\dagger} \cdot \vec{P}_f^{\dagger}$ are non-Hermitians, using them only non-Hermitians Hamiltonians such as H_3 and H_4 , can be constructed. However, we will show that these operators may have applications in some other areas such as quantum finance. In order to see this, we define the potentials

$$U_1(x, y, z) = -\beta^2 \left(\nabla^2 f - \left(\vec{\nabla} f \right)^2 \right) + V_1(x, y, z),$$
(34)

$$U_{2}(x, y, z) = \beta^{2} \left(\nabla^{2} f + \left(\vec{\nabla} f \right)^{2} \right) + V_{2}(x, y, z),$$
(35)

and the non-Hermitians Hamiltonians

$$H_{I} = \beta^{2} \vec{P}_{(f)} \cdot \vec{P}_{(f)} + U_{1}(x, y, z)$$

= $\beta^{2} \left(\vec{P}^{2} + 2i \vec{\nabla} f \cdot \vec{P} \right) + V_{1}(x, y, z),$ (36)

$$H_{II} = \beta^2 \vec{P}^{\dagger}_{(f)} \cdot \vec{P}^{\dagger}_{(f)} + U_2(x, y, z) = \beta^2 \left(\vec{P}^2 - 2i \vec{\nabla} f \cdot \vec{P} \right) + V_2(x, y, z).$$
(37)

On the other hand, let us consider the fundamental equation in quantum finance, the so-called Black-Scholes equation (33)

$$\frac{\partial C}{\partial t} = -\frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2} - rS \frac{\partial C}{\partial S} + rC, \qquad (38)$$

where C is the option price, σ is a constant called the volatility and r is the interest rate [1]. With the change of variable $S = e^x$ we obtain

$$\frac{\partial C}{\partial t} = H_{BS}C,$$

$$H_{BS} = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{\sigma^2}{2} - r\right)\frac{\partial}{\partial x} + r$$
(39)

this non-Hermitian Hamiltonian is called Black-Scholes Hamiltonian. Now, considering the one dimensional case of (37) and identifying

$$\beta^2 = \frac{\sigma^2}{2}, \quad f(x) = \frac{1}{\sigma^2} \left(\frac{\sigma^2}{2} - r\right) x, \quad V_2(x) = r$$

we obtain $H_{II} = H_{BS}$.

One generalized Black-Scholes equation, (see [1]) is given by

$$H_{BSG} = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{\sigma^2}{2} - V(x)\right)\frac{\partial}{\partial x} + V(x).$$
(40)

In this case, considering again the one dimensional case of equation (37) and with

$$\beta^2 = \frac{\sigma^2}{2}, \quad f(x) = \int_0^x du \frac{1}{\sigma^2} \left(\frac{\sigma^2}{2} - V(u)\right), \quad V_2(x) = V(x)$$

we have $H_{II} = H_{BSG}$.

Moreover, the so-called barrier option case has Hamiltonian

$$H_{BSB} = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{\sigma^2}{2} - r\right)\frac{\partial}{\partial x} + V(x).$$
(41)

Again, considering (37) in one dimension and with

$$\beta^2 = \frac{\sigma^2}{2}, \quad f(x) = \frac{1}{\sigma^2} \left(\frac{\sigma^2}{2} - r\right) x, \quad V_2(x) = V(x)$$

we have $H_{II} = H_{BSB}$.

As we have seen, several important Hamiltonians appearing in quantum finance are particular cases of this new version of quantum mechanics.

6 Summary

A quantum mechanics is built by means of a non-Hermitian momentum operator. Moreover, It is shown that using this momentum operator it is possible to construct two Hermitian and two non-Hermitian type of Hamiltonians. Using these Hermitian Hamiltonians we have built, a generalized supersymmetric quantum mechanics with a dual that can be constructed. It also shown that, the non-Hermitian Hamiltonians of this theory may be related to so-called quantum finance Hamiltonian.

References

- [1] B. E. Baaquie, *Quantum Finance*, Cambridge University Press (2004).
- [2] F. Black and M. Scholes, The Pricing Options and Corporate Liabilities, Journal of Political Economy (81), 637-659 (1973).
- [3] C. M. Bender, Introduction to PT-Symmetric Quantum Theory, Contemp. Phys. 46, 277 (2005).
- [4] C. M. Bender, P. D. Mannheim, No-ghost theorem for the fourth-order derivative Pais-Uhlenbeck oscillator model *Phys. Rev. Lett.* **100**, 110402 (2008).
- [5] F. Cooper, A. Khare, U. Sukhatme, Supersymmetry and Quantum Mechanics, *Phys. Rept.* 251, 267-385 (1995).
- [6] P.A. M. Dirac, The Principles of Quantum Mechanics, Oxford (1930).
- [7] P. K. Ghosh, On the construction of a pseudo-Hermitian quantum system with a pre-determined metric in the Hilbert space, J. Phys. A, Math. Theor. 43, 125203 (2010).

- [8] R.C. Merton Theory of Rational Options Pricing, Bell Journal of Economic and Management Science (4), 141-183, (1973).
- [9] S. Shreve, Stochastic Calculus for Finance II: Continuous-Time Models (Springer Finance) (2004).
- [10] T. Tanaka, General aspects of PT-symmetric and P-self-adjoint quantum theory in a Krein space, J. Phys. A 39, 14175 (2006).