## Relativistic Quantum Finance

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#### Abstract

Employing the Klein-Gordon equation, we propose a generalized Black-Scholes equation. In addition, we found a limit where this generalized equation is invariant under conformal transformations, in particular invariant under scale transformations. In this limit, we show that the stock prices distribution is given by a Cauchy distribution, instead of a normal distribution.

## 1 Introduction

The Black-Scholes equation is one of the most useful in finance [1, 2]. This equation can be obtained from different approaches, for example as a limit from Cox-Ross-Rubenstein model or from stochastic calculus. However, Black-Scholes equation is based in some ideal assumptions, for example that there are not arbitrage and that the stock prices follow a normal distribution. Whereby, in some cases the Black-Scholes equation can not provide realistic predictions. This fact has been noted by different authors. Actually, before the Black-Scholes equation was proposed, Mandelbrot noticed that some stock prices do not follow a normal distribution, but a Cauchy distribution [3]. In order to obtain a more realistic Black-Scholes equation, various authors haven been proposed different generalized Black-Scholes equations. For instance using a stochastic volatility [4], multifractal volatility [5], jump

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processes [6], Levy's distributions [7] and fractional differential equations [8].

Interesting, recently diverse mathematical techniques of physics have been applied in finance successfully [9, 10, 11]. For example, quantum mechanics is a natural framework to study financial models with stochastic volatility or stochastic interest rate [9]. Furthermore, statistical mechanics can be used to study financial risk [10] and fluid theory can be employed to study foreign exchange markets [11]. Moreover, statistical arbitrage can be studied with relativistic statistical mechanics [12]. Additionally some financial crashes can be seen as a phase transition [13, 14], it is worth mentioning that when a system is in a phase transition some of its quantities are invariant under scale transformations [15]. In the relation between physics and finance, a remarkably result is that the Black-Scholes equation can be mapped to the free Schrödinger equation [9]. Then, the Black-Scholes formula can be obtained using quantum mechanics. It is worth mentioning that in this mapping the particle mass m is related with the volatility  $\sigma$  in the way  $m \to 1/\sigma^2$ . Thus, an stock price with high volatility is identified with a light weight particle. while a massive particle is identified with a stock price with small volatility. Now, it is well know that when  $\sigma$  is too large the Black-Scholes equation does not make sense. While, when m is very small the Schrödinger equation does not make sense too. In fact, in this last case the quantum mechanics is changed by the relativistic quantum mechanics and the Schrödinger is changed by the Klein-Gordon equation. In the Klein-Gordon equation a new parameter is introduced, the speed of light,  $\tilde{c}$ . When  $\tilde{c} \to \infty$  we obtain the Schödinger equation.

In this paper, in order to obtain a generalized Black-Scholes equation, we relate the Klein-Gordon equation with a new generalized Black-Scholes equation. We show that there is a limit where the generalized Black-Scholes equation is invariant under conformal transformations, in particular invariant under scale transformations. In this limit, we show that the stock prices distribution is given by a Cauchy distribution, instead of a normal distribution. Due that scale invariance is a characteristic of phase transitions [15], we can think that when the modified Black-Scholes equation is invariant under scale transformation the system is near from a phase transition.

This paper is organized as follow: In Section 2, it is shown the mapping between the free Schrödinger equation and the Black-Scholes equation; in Section 3, it is proposed a relativistic Black-Scholes equation; in Section 4, the conformal symmetry is studied; in Section 5, the Cauchy distribution is obtained. Finally, in Section 6 a summary is given.

# 2 Black-Scholes equation and free Schrödinger equation

The Black-Scholes equation is given by

$$\frac{\partial C(S,t)}{\partial t} = -\frac{\sigma^2}{2}S^2 \frac{\partial^2 C(S,t)}{\partial S^2} - rS \frac{\partial^2 C(S,t)}{\partial S} + rC(S,t), \tag{1}$$

where  $\sigma$  is the volatility, S is the stock price, r is the annualized risk-free inters rate and C is the option price. While the free Schrödinger equation is

$$i\hbar\frac{\partial\psi}{\partial\tilde{t}} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2},\tag{2}$$

here *m* is the particle mass,  $\hbar$  is the Planck constant and  $\psi$  is the wave function.

Remarkably, using the mapping

$$\tilde{t} = it, \quad \hbar = 1, \quad m = \frac{1}{\sigma^2}, \quad x = \ln S,$$
(3)

$$\psi(x,t) = e^{-\left(\frac{1}{\sigma^2} \left(\frac{\sigma^2}{2} - r\right)x + \frac{1}{2\sigma^2} \left(\frac{\sigma^2}{2} + r\right)^2 t\right)} C(x,t)$$
(4)

the free Schrödinger equation (2) becomes the Black-Scholes equation (1).

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It is well known that when  $m \to 0$  the Schrödinger equation does note make sense. While, if  $\sigma^2$  is not small the BS equation does not make sense too. In physics the case  $m \to 0$  is not studied in the usual quantum mechanics, but in the relativistic quantum mechanics. In this last theory, the Schrödinger equation is changed by the Klein-Gordon equation

$$-\frac{\hbar^2}{\tilde{c}^2}\frac{\partial^2\psi(x,t)}{\partial\tilde{t}^2} + \frac{\partial^2\psi(x,t)}{\partial x^2} - m^2\tilde{c}^2\psi(x,t) = 0,$$
(5)

where  $\tilde{c}$  is the light speed. Notice that when  $m \to 0$ , the Klein-Gordon equation makes sense. Furthermore, it can be shown that when  $\tilde{c} \to \infty$  the Klein-Gordon equation becomes the Schrödinger equation.

Now, using the mapping

$$\tilde{t} = it, \quad \hbar = 1, \quad m = \frac{1}{\sigma^2}, \qquad x = \ln S,$$
(6)

$$\tilde{c}^2 = q, \qquad \psi(x,t) = e^{-\left[\frac{1}{\sigma^2}\left(\frac{\sigma^2}{2} - r\right)x + \left(\frac{1}{2\sigma^2}\left(\frac{\sigma^2}{2} + r\right)^2 - \frac{q}{\sigma^2}\right)t\right]}C(x,t) \quad (7)$$

the Klein-Gordon equation becomes

$$\frac{1}{q}\frac{\partial^2 C(S,t)}{\partial t^2} + \left(\frac{2}{\sigma^2} - \frac{1}{q\sigma^2}\left(\frac{\sigma^2}{2} + r\right)^2\right)\frac{\partial C(S,t)}{\partial t} + S^2\frac{\partial^2 C(S,t)}{\partial S^2} + \frac{2r}{\sigma^2}S\frac{\partial C(S,t)}{\partial S} + \left[\frac{1}{4q\sigma^4}\left(\frac{\sigma^2}{2} + r\right)^4 - \frac{2r}{\sigma^2}\right]C(S,t) = 0.$$
(8)

This equation can be written as

$$\frac{\sigma^2}{2q} \frac{\partial^2 C(S,t)}{\partial t^2} + \left(1 - \frac{1}{2q} \left(\frac{\sigma^2}{2} + r\right)^2\right) \frac{\partial C(S,t)}{\partial t}$$
$$= -\frac{\sigma^2}{2} S^2 \frac{\partial^2 C(S,t)}{\partial S^2} - rS \frac{\partial C(S,t)}{\partial S} + rC(S,t). \tag{9}$$

Furthermore, if we take the limit  $q \to \infty$ , in this last equation we arrive to the Black-Scholes equation (1). Then, the equation (8) is a generalized Black-Scholes equation, which has the new parameter q.

### 4 Conformal symmetry

The usual Black-Scholes equation is invariant under the Schrödinger group [16].

In orden to understand the symmetries for the equation (8) we take the coordinates

$$z = \ln S + i\sqrt{q}t. \tag{10}$$

Using this coordinates, the modified Black-Scholes equation (8) can be written as

$$4\frac{\partial^2 C}{\partial z \partial \bar{z}} + 2\left(\bar{A}\frac{\partial C}{\partial z} + A\frac{\partial C}{\partial \bar{z}}\right) + \left(A\bar{A} - \frac{q}{\sigma^4}\right)C = 0, \tag{11}$$

where

$$A = -\frac{1}{\sigma^2} \left[ \left( \frac{\sigma^2}{2} - r \right) - \frac{i}{2\sqrt{q}} \left( \left( \frac{\sigma^2}{2} + r \right)^2 - 2q \right) \right].$$
(12)

We can see that the equation (11) is invariant under the following transformations

$$z' = e^{i\alpha}z, \qquad \alpha = \text{constant},$$
 (13)

$$C'(z,\bar{z}') = e^{\frac{1}{2}[Az(1-e^{i\alpha})+\bar{A}\bar{z}(1-e^{-i\alpha})]}C(z,\bar{z}).$$
(14)

We have a special case, in fact when

$$\frac{q}{\sigma^4} \ll \bar{A}A,\tag{15}$$

that is

$$0 << \frac{1}{4q} \left(\frac{\sigma^2}{2} + r\right)^4 + 2r\sigma^2, \tag{16}$$

the equation (11) becomes

$$4\frac{\partial^2 C}{\partial z \partial \bar{z}} + 2\left(\bar{A}\frac{\partial C}{\partial z} + A\frac{\partial C}{\partial \bar{z}}\right) + A\bar{A}C = 0.$$
(17)

This last equation is invariant under the conformal symmetry

$$z' = z'(z), (18)$$

$$C'(z',\bar{z}') = e^{\frac{1}{2} \left[ A(z-z') + \bar{A}(\bar{z}-\bar{z}') \right]} C(z,\bar{z}).$$
(19)

where

$$\frac{\partial z'(z)}{\partial \bar{z}} = 0, \tag{20}$$

namely z'(z) is an analytic function. Then, every analytic function provides a symmetry for the equation (17).

Notice that the equation (20) implies that at first order any infinitesimal transformation can be expressed as

$$z' = z + \epsilon(z), \qquad \epsilon(z) = \sum_{-\infty}^{\infty} \epsilon_n z^{n+1}, \qquad \epsilon_n = \text{constant.}$$
 (21)

Now, using the equation (19), we find

$$C'(z',\bar{z}') \approx C(z',\bar{z}') - \left[\epsilon(z')\left(\frac{A}{2} + \frac{\partial}{\partial z'}\right) + \bar{\epsilon}(\bar{z}')\left(\frac{\bar{A}}{2} + \frac{\partial}{\partial \bar{z}'}\right)\right]C(z',\bar{z}'), (22)$$

that is

$$\delta C(z', \bar{z}') = C'(z', \bar{z}') - C(z', \bar{z}')$$
  

$$\approx -\left[\epsilon(z')\left(\frac{A}{2} + \frac{\partial}{\partial z'}\right) + \bar{\epsilon}(\bar{z}')\left(\frac{\bar{A}}{2} + \frac{\partial}{\partial \bar{z}'}\right)\right]C(z, \bar{z}). \quad (23)$$

Furthermore, using the equation (21) the option price transforms as

$$\delta C(z,\bar{z}) = \sum_{-\infty}^{\infty} \left( \epsilon_n l_n + \bar{\epsilon}_n \bar{l}_n \right) C(z',\bar{z}'), \qquad (24)$$

where

$$l_n = -z^n \left(\frac{A}{2} + \frac{\partial}{\partial z}\right), \qquad \bar{l}_n = -\bar{z}^n \left(\frac{\bar{A}}{2} + \frac{\partial}{\partial \bar{z}}\right). \tag{25}$$

These operator satisfy the Witt algebra

$$\begin{bmatrix} l_n, l_k \end{bmatrix} = (n-k) l_{n+k}, \begin{bmatrix} \bar{l}_n, \bar{l}_k \end{bmatrix} = (n-k) \bar{l}_{n+k}, \begin{bmatrix} \bar{l}_n, l_k \end{bmatrix} = 0.$$
 (26)

Notably, the usual Black-Scholes equation has only a finite number of symmetries, but the equation (17) has an infinity number of symmetries.

In particular, if  $\lambda$  is a real number, the equation (17) is invariant under the scale transformations

$$z' = \lambda z, \tag{27}$$

$$C'(z', \bar{z}') = e^{\frac{1}{2}(1-\lambda)(Az+\bar{A}\bar{z})}C(z, \bar{z}),$$
 (28)

which can be written as

$$S' = S^{\lambda}, \tag{29}$$

$$t' = \lambda t, \tag{30}$$

$$C'(S',t') = C'(S^{\lambda},\lambda t) = S^{\frac{(\lambda-1)}{\sigma^2} \left(\frac{\sigma^2}{2} - r\right)} e^{\frac{(\lambda-1)}{2\sigma^2} \left(\left(\frac{\sigma^2}{2} + r\right)^2 - 2q\right)t} C(S,t).(31)$$

In general, the equation (8) is not invariant under scale transformations, however in the limit (16) this symmetry is obtained. This phenomenon is well known in phase transitions, in fact the scale invariance is a characteristic of phase transitions [15]. Then, we can think that when the equation (16) is satisfied, the system is near from a phase transition. Remarkably, when a system is near from a phase transition there are fluctuations of all scales. This phenomenon happened in some financial crashes [13, 14].

### 5 Cauchy distribution

Using the coordinates x, t and the function  $\psi$  defined in (7), the equation (17) can be written as

$$\frac{1}{q}\frac{\partial^2\psi(x,t)}{\partial t^2} + \frac{\partial^2\psi(x,t)}{\partial x^2} = 0.$$
(32)

Now, imposing the initial condition

$$\psi(x,0) = f(x),\tag{33}$$

the solution for the equation (32) is given by

$$\psi(x,t) = \int_{-\infty}^{\infty} d\zeta G(x-\zeta,t) f(\zeta), \qquad (34)$$

where

$$G(x - \zeta, t) = \frac{1}{\pi} \frac{\sqrt{qt}}{(x - \zeta)^2 + qt^2}.$$
(35)

Notice that this last function is the Cauchy distribution. Using the mapping (4), we obtain the option price

$$C(x,t) = \int_{-\infty}^{\infty} d\zeta K(x-\zeta,t)C(\zeta,0), \qquad (36)$$

here

$$K(x-\zeta,t) = \frac{e^{\frac{1}{\sigma^2} \left[ \left( \frac{1}{2} \left( \frac{\sigma^2}{2} + r \right)^2 - q \right) t + \left( \frac{\sigma^2}{2} - r \right) (x-\zeta) \right]}{\pi} \frac{\sqrt{qt}}{(x-\zeta)^2 + qt^2}.$$
 (37)

The Cauchy distribution was first proposed by B. Mandelbrot as a distribution for stock price [3].

#### 6 Summary

Employing the Klein-Gordon equation, we proposed a generalized Black-Scholes equation. We found a limit where the generalized equation is invariant under conformal transformations, in particular invariant under scale transformations. In this limit, we shown that the stock prices distribution is given by a Cauchy distribution, instead of a normal distribution.

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