

Relativistic conformal symmetry of neural field propagation in the brain

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Abstract

In this paper, we address a neural field equation that characterizes spatio-temporal propagation of a neural population pulse. Due that the human brain is a complex system whose constituents interaction give rise to fundamental states of consciousness and behavior, it is crucial to gain insight into its functioning even at relativistic scales. To this end, we study the action of the relativistic conformal group on the accounted neural field propagation equation. In particular, we obtain an exact solution for the field propagation equation when the space-time is 3 or 4 dimensional. Furthermore, in the 4 dimensional

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case and the large distance limit, it is shown that the neural population pulse becomes a Yukawa potential.

1 Introduction

In the last decades, human brain studies have attracted considerable attention due to prevalent occurrence of brain-related disorders with an aging population. From an economic perspective, the impact of such phenomenon has been emphasized by international health organizations that predict increasing healthcare spending accompanied with a reduction of life quality for the ill in both developing and developed nations [1].

In order to gain understanding of the human brain functioning, modern experimental approaches rely on different imaging techniques as in the case of the electroencephalogram (EEG), which encompasses cortical electrical activity aggregated over scales much larger than individual neurons. In this respect, recent experimental and theoretical studies [2, 3, 4] support that a neural population pulse $\phi_a(\vec{x}, t)$ of axonal signals propagates according to the neural field propagation equation

$$\left(\frac{1}{\gamma_a^2} \frac{\partial^2}{\partial t^2} + \frac{2}{\gamma_a} \frac{\partial}{\partial t} + 1 - r_a^2 \nabla^2 \right) \phi_a(\vec{x}, t) = Q_a(\vec{x}, t), \quad (1)$$

where r_a is the mean range of axons, v_a is the wave velocity, $\gamma_a = \frac{v_a}{r_a}$ is the temporal damping coefficient and $Q_a(\vec{x}, t)$ is the mean firing rate, $a = e, i$ denotes neural excitatory and inhibitory activity, respectively. Moreover, a good approximation for a neural field firing rate according to [2, 3, 4] is given by

$$Q_a(\vec{x}, t) = Q_{a(max)} S_a[V_a(\vec{x}, t); \sigma_a] \quad (2)$$

where $Q_{a(max)}$ is the maximum firing rate, S_a is the rate-voltage response function and σ_a is the population standard deviation of the soma voltage $V_a(\vec{x}, t)$ relative to the firing threshold.

From a geometrical point of view, the importance of conformal transformations is emphasized by recent studies addressing geometric preserving mappings of the human cortex [5]. In addition, conformal symmetry has

been useful for studying physical systems as in the case of the classical free particle, the free Schrödinger equation and the Fick diffusion equation. In particular, it has been shown that these systems are invariant under the non-relativistic conformal group [6, 7, 8, 9, 10]. In fact, the relativistic conformal symmetry is the largest relativistic symmetry enabling a correspondence between a gravity theory and a gauge theory [11]. This correspondence plays a crucial role in theoretical physics, while also emphasizes the important role of conformal geometry.

In this paper we show that the neural field propagation equation is invariant under the relativistic conformal group. When the space-time is 3 or 4 dimensional, an exact solution for such equation is introduced. Furthermore, in the case $d = 4$ it is shown that, in the large distant limit, the neural population pulse becomes a Yukawa-like potential.

This paper is organized as follow: in section 2 we provide a brief overview of relativistic conformal symmetry. In section 3 we show that the neural field propagation equation is invariant under the relativistic conformal group. In section 4 a summary is given.

2 Relativistic conformal Symmetry

If $\eta_{\mu\nu}$ is the Minkowski metric, the line element is given by

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \quad x^\mu = (x^0 = ct, x^1, \dots, x^d), \quad (3)$$

where c is the speed of light. Now, the coordinate transformation

$$x'^\mu = x'^\mu(x) \quad (4)$$

is conformal if it satisfies

$$ds'^2 = \eta_{\mu\nu} dx'^\mu dx'^\nu = \Omega(x) \eta_{\mu\nu} dx^\mu dx^\nu. \quad (5)$$

It can be shown that the following coordinate transformations

$$x'^\mu = \lambda x^\mu, \quad (6)$$

$$x'^\mu = x^\mu + b^\mu, \quad (7)$$

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \quad \eta'^{\mu\nu} = \eta^{\mu\alpha} \Lambda_{\alpha\beta} \eta^{\beta\nu}, \quad (8)$$

$$x'^\mu = \frac{x^\mu + a^\mu x^2}{1 + a \cdot x + a^2 x^2}, \quad (9)$$

satisfy the equation (5). Here $\lambda, b^\mu, a^\mu, \Lambda_{\mu\nu}$ are constants. Equation (6) represents scale transformation; equation (7) represents translations on space-time; equation (8) represents Lorentz transformations and equation (9) represents the special conformal transformations.

It is worth mentioning that the massless relativistic particle equation

$$\partial_\mu \partial^\mu \psi = 0. \quad (10)$$

is invariant under the relativistic conformal group (6)-(9). In this case, under special conformal transformation (9), the field ψ transforms as

$$\psi'(x'^\mu) = \Omega^{\frac{d-2}{2}}(x)\psi(x^\mu), \quad \Omega(x) = 1 + a \cdot x + a^2 x^2. \quad (11)$$

In next section, it will be shown that the neural field propagation equation (1) is invariant under conformal transformations (6)-(9).

3 Symmetries for the neural population pulse

The neural field propagation equation (1) is invariant under translations on space-time (7), where the neural population pulse and firing rate transform as

$$\phi'_a(\vec{x}', t') = e^{\frac{-\gamma_a a^0}{v_a}} \phi_a(\vec{x}, t), \quad (12)$$

$$Q'_a(\vec{x}', t') = e^{\frac{-\gamma_a a^0}{v_a}} Q_a(\vec{x}, t). \quad (13)$$

Then, the neural population pulse and firing rate are not scalar under translations on space-time.

Furthermore, the neural field propagation equation (1) is invariant under scale transformations

$$t' = \lambda t, \quad \vec{x}' = \lambda \vec{x}, \quad (14)$$

where the neural population pulse and firing rate transform as

$$\phi'_a(\vec{x}', t') = e^{\frac{v_a}{r_a} t(1-\lambda)} \phi_a(\vec{x}, t), \quad (15)$$

$$Q'_a(\vec{x}', t') = \frac{e^{v_a(1-\lambda)t} r_a}{\lambda^2} Q_a(\vec{x}, t). \quad (16)$$

These two last transformations are unusual, although interesting as the neural field propagation equation is not obviously invariant under scale transformations.

In addition, the neural field propagation equation (1) is invariant under Lorentz transformations

$$\begin{aligned} t' &= \gamma \left(t - \frac{vx_1}{v_a^2} \right), & \gamma &= \frac{1}{\sqrt{1 - \frac{v^2}{v_a^2}}}, \\ x'_1 &= \gamma (x_1 - vt), \\ x'_2 &= x_2, \\ &\vdots \\ x'_d &= x_d, \end{aligned}$$

here the neural population pulse and firing rate transform as

$$\phi'_a(\vec{x}', t') = e^{\frac{v_a}{r_a} t(1-\gamma)} e^{\frac{\gamma v}{r_a v_a} x_1} \phi_a(\vec{x}, t), \quad (17)$$

$$Q'_a(\vec{x}', t') = e^{\frac{v_a t}{r_a}} e^{-\frac{v_a}{r_a} \gamma \left(t - \frac{vx_1}{v_a^2} \right)} Q_a(\vec{x}, t). \quad (18)$$

Notice that in this case v_a plays the role of the light speed c . The transformations (17) and (18) are unusual in a physical system, although they are interesting because the neural field propagation equation is not obviously invariant under the Lorentz transformation. Indeed, it is remarkable that electrical brain pulses exhibit relativistic symmetry.

Moreover, if it is taken $x^0 = v_a t$, the neural field propagation equation (1) is invariant under special conformal transformations (9), where the neural population pulse and firing rate transform as

$$\begin{aligned} \phi'_a(\vec{x}', t') &= \Omega^{\frac{d-2}{2}} e^{-\frac{v_a}{r_a \Omega} \left(t(1-\Omega) + \frac{a^0 x^2}{v_a} \right)} \phi_a(\vec{x}, t), \\ Q'_a(\vec{x}', t') &= \Omega^{\frac{d+2}{2}} e^{-\frac{v_a}{r_a \Omega} \left(t(1-\Omega) + \frac{a^0 x^2}{v_a} \right)} Q_a(\vec{x}, t). \end{aligned}$$

Notice that in the conformal field theory, under special conformal transformations, a scalar field transform as (11). Then, these two last transformations are unusual in invariant systems under a special conformal symmetry. However, it has recently been shown that fields which are transformed under a

special conformal transformation in an unusual way appear in different contexts. For instance, in the so-called logarithmic conformal field theory, the fields are transformed in an unusual way [14, 15]. This theory is applied to study a version of Gravity/Gauge correspondence [16].

4 Wave equation

Now, if we take $\phi_a(\vec{x}, t) = e^{-\gamma_a t} \psi_a(\vec{x}, t)$ the equation (1) becomes

$$r_a^2 e^{-\gamma_a t} \left(\frac{1}{v_a^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \psi_a(\vec{x}, t) = Q_a(\vec{x}, t) \quad (19)$$

which is equivalent to

$$\left(\nabla^2 - \frac{1}{v_a^2} \frac{\partial^2}{\partial t^2} \right) \psi_a(\vec{x}, t) = -\frac{e^{\gamma_a t}}{r_a^2} Q_a(\vec{x}, t). \quad (20)$$

This last equation is the no-homogenous wave equation.

4.1 Case $d = 3$

When $d = 3$, the solution for equation (20) is given by [13]

$$\psi_a(\vec{x}, t) = \frac{1}{4\pi r_a^2} \int d^2 x' v_a dt' \frac{2e^{\gamma_a t'}}{\sqrt{v_a^2 (t-t')^2 - (\vec{x} - \vec{x}')^2}} Q_a(\vec{x}', t').$$

Then, the neural population pulse is

$$\phi_a(\vec{x}, t) = \frac{v_a}{2\pi r_a^2} \int d^2 x' dt' \frac{e^{-\gamma_a (t-t')}}{\sqrt{v_a^2 (t-t')^2 - (\vec{x} - \vec{x}')^2}} Q_a(\vec{x}', t').$$

Notice that in this case the following equation

$$\frac{(\vec{x} - \vec{x}')^2}{(t - t')^2} \leq v_a^2 \quad (21)$$

has to be satisfied.

4.2 Case $d = 4$

When $d = 4$, the solution for equation (20) is given by [13]

$$\psi_a(\vec{x}, t) = \frac{1}{4\pi r_a^2} \int d^3x' \frac{e^{\frac{v_a}{r_a} \left(t - \frac{|\vec{x} - \vec{x}'|}{v_a} \right)}}{|\vec{x} - \vec{x}'|} Q_a(\vec{x}', t_R),$$

where

$$t_R = t - \frac{|\vec{x} - \vec{x}'|}{v_a}$$

is the retarded time. Then, the neural population pulse is

$$\phi_a(\vec{x}, t) = \frac{1}{4\pi r_a^2} \int d^3x' \frac{e^{-\frac{|\vec{x} - \vec{x}'|}{r_a}}}{|\vec{x} - \vec{x}'|} Q_a(\vec{x}', t_R). \quad (22)$$

4.3 Yukawa Potential

When $|\vec{x}'| \ll |\vec{x}|$, the following expressions

$$|\vec{x}' - \vec{x}| \approx |\vec{x}| - \hat{x} \cdot \vec{x}', \quad \frac{1}{|\vec{x}' - \vec{x}|} \approx \frac{1}{|\vec{x}|} \quad (23)$$

are obtained. Then, the neural population pulse (22) becomes

$$\phi_a(\vec{x}, t) = \frac{q_a(\vec{x}, t) e^{-\frac{|\vec{x}|}{r_a}}}{4\pi r_a^2 |\vec{x}|} \quad (24)$$

here

$$q_a(\vec{x}, t) = \int d\vec{x}' e^{\frac{\hat{x} \cdot \vec{x}'}{r_a}} Q_a(\vec{x}', t_R), \quad t_R = t - \frac{|\vec{x}|}{v_a} + \frac{\hat{x} \cdot \vec{x}'}{v_a}.$$

The neural pulse (24) is a Yukawa-like potential [13].

5 Summary

We have shown that a neural field propagation equation, used for targeting large-scale responses of the human brain, is invariant under the relativistic conformal group. When the space-time is 3 or 4 dimensional, an exact solution for the neural field propagation equation was reported. Furthermore, in the case $d = 4$ it was shown, that in the large distant limit, the neural population pulse becomes a Yukawa-like potential.

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