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# Bulk viscosity in 2D relativistic fluids: the effects of temperature and modifications to the Rayleigh-Brillouin spectrum

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**Abstract.** The dependence of the bulk viscosity with the relativistic parameter  $z = kT/mc^2$ , obtained through the complete Boltzmann equation [1, 2], is thoroughly analyzed. A complete and rigorous examination of the relevant non-relativistic and ultra-relativistic limits is carried out in the case of a hard disk model and compared with the results obtained in the relaxation time approximation. The modifications of the non-vanishing bulk viscosity to the Rayleigh-Brillouin is briefly discussed.

## 1. Introduction

The study of relativistic bidimensional fluids is a relevant and interesting topic which is still in need of extensive analysis. Additional to its importance in the modelling of axisymmetric systems, for example a charged fluid in the presence of a magnetic field, orbiting gases, or accretion in gravitational potentials, the interest in the study of bidimensional systems in relativistic scenarios has seen a substantial increase due to the new generation of thin materials and their several applications. Also, relativistic fluids still pose a challenge, plagued with unanswered questions and topics of intense debate even in very fundamental issues, numerical simulations in low dimensionality have been one of the most valuable assets in order to corroborate theoretical predictions.

The establishment of relativistic transport coefficients in a special relativistic framework from the complete Boltzmann equation has only been addressed recently in the 2D scenario. Indeed, in a separate publication (see Ref. [3]) the constitutive equations are established and the transport coefficients expressed in terms of collision integrals which, for a hard disks model, can be numerically evaluated. In particular, the bulk viscosity is expressed in terms of the integral

$$\mathcal{I}(z) = \int_{\frac{2}{z}}^{\infty} e^{-(x-\frac{2}{z})} \left( \frac{1}{x} + \frac{3}{x^2} + \frac{3}{x^3} \right) (z^2 x^2 - 4)^{5/2} dx \quad (1)$$

whose dependence on  $z$  is not trivial. Moreover, its convergence in the non-relativistic ( $z \rightarrow 0$ ) and ultrarelativistic ( $z \rightarrow \infty$ ) limits is not completely justified. This is precisely part of the goal of the present work, the formal establishment of the behaviour of Eq. (1) in the non-relativistic and ultrarelativistic limits. Also, in order to address the effect of a non-zero bulk



viscosity of the gas for the complete range of  $z$  we study the corresponding modifications on the Rayleigh-Brillouin spectrum.

In order to accomplish such a task, the rest of the work is organized as follows. In sections 2 and 3 we briefly outline the procedure carried out in Ref. [3] in order to establish the analytical expression for the bulk viscosity in the specific case of a hard disks model. Section 4 is devoted to the analysis of the integral defined in Eq. (1) in the non relativistic and ultrarelativistic limits, while section 5 addresses the modification that a non-vanishing bulk viscosity has on the Brillouin peaks for a light scattering spectrum. The discussion of the results and final remarks are included in section 6.

## 2. Boltzmann equation and Chapman-Enskog approximation

In this section, we provide a brief description of the procedure leading to the function  $\mathcal{I}(z)$  in Eq. (1), whose relevant limits are explored in the main part of this work. The complete calculation will be published elsewhere (a preprint can be found in Ref. [3]) together with the calculation of the rest of the relevant coefficients, to which the reader is referred to for further details.

The starting point is the relativistic Boltzmann equation [2] in a flat spacetime with a  $(+---)$  metric which reads

$$v^\alpha \frac{\partial}{\partial x^\alpha} f(x^\nu, v^\nu) = \int \int (\tilde{f}\tilde{f}_1 - ff_1) F \sigma d\chi dv_1^*, \quad (2)$$

where  $f(x^\nu, v^\nu)$  is the one particle distribution function,  $F$  is the invariant flux,  $\gamma = u^\mu v_\mu / c^2$ , with  $v^\mu$  being the molecular 3-velocity measured in an arbitrary frame,  $u^\mu$  is the fluid's 3-velocity and  $\sigma$  and  $\chi$  are the corresponding scattering cross section and solid angle in 2D. The local equilibrium solution to Eq. (2) is given by a bidimensional Maxwell-Jüttner distribution, that is

$$f^{(0)}(v^\nu) = \frac{ne^{\frac{1}{z}}}{2\pi c^2 z (1+z)} e^{-\frac{u^\alpha v_\alpha}{zc^2}} \quad (3)$$

where  $u^\alpha$  corresponds to the fluid's 3-velocity. With such distribution one can establish statistical definition for the state variables:  $n$  (number density),  $u^\nu$  (hydrodynamic 3-velocity) and  $\varepsilon$  (internal energy). Considering the particles frame (Eckart's frame), one has

$$n = \int f^{(0)}(v^\nu) \gamma dv^*, \quad (4)$$

$$nu^\nu = \int f^{(0)}(v^\nu) v^\nu dv^*, \quad (5)$$

$$n\varepsilon = mc^2 \int f^{(0)}(v^\nu) \gamma^2 dv^*, \quad (6)$$

from which one can obtain

$$n\varepsilon = nmc^2 g(z) \quad \text{with} \quad g(z) = \frac{2z^2 + 2z + 1}{z(z+1)}. \quad (7)$$

These quantities, in the absence of dissipation, follow Euler's equations. In order to establish the first order in the gradients distribution, the Chapman-Enskog solution to Eq. (2) reads

$$f(v^\nu) = f^{(0)}(v^\nu) (1 + \phi(v^\nu)), \quad (8)$$

where the first order correction to the local equilibrium distribution function  $\phi(v^\nu)$  is given by the solution of the linearized Boltzmann equation.

$$f^{(0)} \int \int f^{(0)}(v_1) \left( \tilde{\phi}_1 + \tilde{\phi} - \phi_1 - \phi \right) F \sigma d\chi dv_1^* = v^\alpha \frac{\partial f^{(0)}(v^\nu)}{\partial x^\alpha}. \quad (9)$$

The dissipative fluxes, which arise from such deviation from equilibrium, are found as moments of  $f^{(0)}(v) \phi(v^\nu)$ . In particular, for the energy momentum tensor, which is the focus of the present work, one has

$$\pi^{\mu\nu} = m h^{\mu\alpha} h^{\nu\beta} \int f^{(0)}(v) \phi(v^\nu) v_\alpha v_\beta d^*v. \quad (10)$$

Here  $h^{\alpha\beta} = \eta^{\alpha\beta} - u^\alpha u^\beta / c^2$  is the spatial projector corresponding to the (2+1) representation where  $u^\alpha$  corresponds to the temporal direction in the comoving frame.

The details of the calculation that follows can be found in Ref.[3]. However the authors consider it worthwhile to point out here the particular step to which the occurrence of a finite bulk viscosity in the relativistic regime, opposed to the zero value obtained for non-relativistic gases, can be traced down to. Equation (9) is a linear integral equation, whose solution is a superposition of the homogeneous and a particular solutions. It is from the latter that the driving terms for the deviation will appear as the gradients of the state variables. In such a term one finds, for the part that depends on the velocity gradient and will thus lead to viscous dissipation,

$$\left( v^\alpha f_{,\alpha}^{(0)} \right)_{visc.} = f^{(0)} \left\{ \left( \frac{v^\beta u_\beta}{c^2} \right) \left( \frac{1}{n} u^\alpha n_{,\alpha} + \frac{1}{T} \left( \frac{\gamma}{z} - g(z) \right) u^\alpha T_{,\alpha} \right) + \frac{v_\mu v^\beta h_\beta^\alpha}{z c^2} u_{,\alpha}^\mu \right\} \quad (11)$$

where we have introduced the usual decomposition  $v^\alpha = v^\beta h_\beta^\alpha + (v^\beta u_\beta / c^2) u^\alpha$ , in this case in a (2+1) framework. Notice that the proper time derivatives of the number density and temperature are coupled with the velocity gradient through Euler's equations

$$u^\alpha n_{,\alpha} = -n u_{,\alpha}^\alpha \quad \text{and} \quad u^\alpha T_{,\alpha} = -k_p(z) T u_{,\alpha}^\alpha \quad (12)$$

where  $k_p(z) = k/C_n = (z+1)/(z(g(z)+2(z+1)))$ . Also, the term  $u^\alpha u_{,\alpha}^\mu$  is not included in Eq. (11) since the momentum equation in the local equilibrium case couples it only with the pressure gradient.

Introducing Eqs. (12) in Eq. (11) and splitting the velocity gradient in antisymmetric, symmetric traceless parts and the trace times an identity tensor one can write

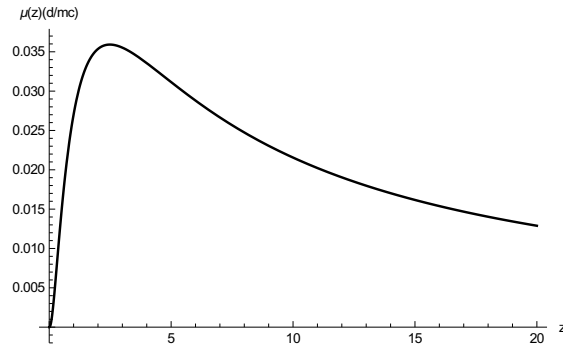
$$\left( v^\alpha f_{,\alpha}^{(0)} \right)_{visc.} = \left[ \left( \frac{1}{2} - k_p(z) \right) \frac{\gamma_k^2}{z} + (k_p(z) g(z) - 1) \gamma_k - \frac{1}{2z} \right] u_{,\nu}^\nu - \frac{v^\mu v^\alpha}{z c^2} \hat{\sigma}_{\mu\alpha}, \quad (13)$$

where  $\hat{\sigma}_{\mu\alpha}$  is the symmetric traceless component of  $u_{\mu,\alpha}$ . Notice that, in the non-relativistic case, the term in brackets vanishes since in such a limit  $z \rightarrow 0$ ,  $\gamma \rightarrow 1$  and thus  $g(z) \sim z^{-1} + 1$  and  $k_p(z) \sim 1 - 2z$ . Due to this fact, the bulk viscosity is zero for the monoatomic ideal gas at low temperatures. However, as will be verified in the next section, is non-zero for  $z \neq 0$  and increases rapidly for  $0 < z < 1$ .

Once the driving term given by Eq. (13) is substituted in Eq. (9), a constitutive equation for the scalar part of the Navier tensor can be found. Such a relation is written as

$$\pi_\mu^\mu = 2\mu u_{,\alpha}^\alpha. \quad (14)$$

Following the standard procedure (see for example Refs. [1, 4]) one can obtain a first approximation for  $\mu$ . The steps are carefully detailed in Ref. [3], here we only quote the result:



**Figure 1.** The dimensionless bulk viscosity as a function of the parameter  $z$  in the relativistic scenario.

$$\mu = \frac{4z^7}{(1+z)^2} \frac{mc^2}{(2z^2+4z+1)^2} [\gamma_k^2, \gamma_k^2]^{-1}, \quad (15)$$

where

$$[H, G] = -\frac{1}{n^2} \int \mathcal{C}(H) G f^{(0)} d^*v \quad (16)$$

is the collision bracket, which satisfies

$$[H, G] = -\frac{1}{4n^2} \int (H' + H'_1 - H - H_1) (G' + G'_1 - G - G_1) f^{(0)} f_*^{(0)} F \sigma(\chi) d\chi dv_1^* dv^* d^*v. \quad (17)$$

In the next section, the behavior of  $\mu$  as a function of  $z$  for a hard disks model will be described.

### 3. The temperature dependence of the viscosity for a hard disk gas

The so-called collision integrals defined in Eq. (16) depend on a molecular interaction model for the system. The simplest case in a bidimensional scenario consists on a hard disk model for which the scattering cross section is given by

$$\sigma(\chi) = \frac{d}{2} \left| \sin\left(\frac{\chi}{2}\right) \right|.$$

In such a case, as is shown in Appendix C of Ref. [3], the relevant collision integral is given by

$$[\gamma^2, \gamma^2] = \frac{2cz^3d}{15} \frac{1}{(z+z^2)^2} \mathcal{I}(z), \quad (18)$$

where the integral  $\mathcal{I}(z)$  is given in Eq. (1). Thus, one can write the bulk viscosity for the system as

$$\mu(z) = \frac{30mc}{d} \frac{z^6}{(2z^2+4z+1)^2} \mathcal{I}(z)^{-1}. \quad (19)$$

In the next section, the non-relativistic limit of this expression will be carefully addressed. However, it can be seen at this point, by inspection of Fig. 1 that  $\mu$  vanishes at  $z = 0$ . It is important to notice that for  $z$  finite,  $\mu \neq 0$  and moreover, it increases very rapidly with temperature.

Bulk viscosity for the monoatomic ideal gas is thus non-zero as long as  $T \neq 0$  and becomes relevant for some range of values of  $z$ . The fact that it reaches a maximum value, which can be

numerically obtained as  $\mu_{max} \sim \mu(2.4886) \sim 0.0359$ , and then decreases for higher temperatures is also found in the three dimensional case and deserves a closer analysis. One can then conclude that for a limited range of temperature, for each gas, viscosity is enhanced. This could lead to faster damping/enhancing of instabilities.

#### 4. Non-relativistic and ultrarelativistic limits of $\mu(z)$

In this section, a formal proof of the non-relativistic and ultrarelativistic limits of the expression for  $\mu$  obtained in Ref. [3] and quoted in the previous section is detailed. As mentioned above, inspection of Fig. 1 points towards the bulk viscosity approaching zero in the non-relativistic and ultra-relativistic limits. This behavior is proven separately for each case. In particular, the non-relativistic limit of this and other transport coefficients is far from trivial since the lower integration limit tends to infinity and thus the usual techniques cannot be applied.

We thus begin by addressing the low temperature, non-relativistic, case. Let us start defining the function

$$h(x, z) = e^{-(x-\frac{2}{z})} \left( \frac{1}{x} + \frac{3}{x^2} + \frac{3}{x^3} \right) (z^2 x^2 - 4)^{5/2}, \quad (20)$$

which corresponds to the integrand in  $\mathcal{I}(z)$  and an auxiliary function

$$k(x, z) = e^{-(x-\frac{2}{z})} \left( \frac{1}{x^2} \right) (z^2 x^2 - 4)^3, \quad (21)$$

both well defined on  $A = (x, z) \in \mathbb{R}^2, |xz \geq 2, z > 0$ .

**Proposition 1.** For all  $(x, z) \in A$  and  $z < 1$ , it is verified  $0 < k(x, z) \leq h(x, z)$ .

*Proof.* The equality holds for  $x = 2/z$ . For  $x > 2/z$  we claim

$$e^{-(x-\frac{2}{z})} \left( \frac{1}{x^2} \right) (z^2 x^2 - 4)^3 < e^{-(x-\frac{2}{z})} \left( \frac{1}{x} + \frac{3}{x^2} + \frac{1}{x^3} \right) (z^2 x^2 - 4)^{5/2}$$

which is equivalent to

$$(z^2 x^2 - 4)^{1/2} < \left( x + 3 + \frac{3}{x} \right).$$

If we define  $x = 2\alpha/z$  for  $\alpha \in (1, \infty)$ , last inequality can be written as

$$2z(\alpha^2 - 1)^{1/2} < 2\alpha + 3z + \frac{3z^2}{2\alpha}$$

which holds for any  $\alpha > 1$  and for  $z \in (0, 1)$ . □

**Theorem 1.**  $\mu(z) \rightarrow 0^+$  as  $z \rightarrow 0^+$ .

*Proof.* Since  $\mu \geq 0$  for  $z > 0$ , we can establish that the limit vanishes by simply upper-bounding the function by an auxiliary real-valued analytic function that tends to zero in such a limit.

From Proposition 1 we have

$$0 < \int_{\frac{2}{z}}^{\infty} k(x, z) dx \leq \int_{\frac{2}{z}}^{\infty} h(x, z) dx, \quad (22)$$

where the integral on the left hand side can be computed as

$$\begin{aligned} \int_{\frac{2}{z}}^{\infty} k(x, z) dx &= 8(-4z + 2z^2 - 2z^3 + 3z^4 + 6z^5 + 3z^6) + 64e^{2/z} \int_{2/z}^{\infty} \frac{e^{-t}}{t} dt, \\ &= 96z^5 + \mathcal{O}(z^6) \end{aligned} \quad (23)$$

for  $z \in (0, \epsilon)$ , where  $0 < \epsilon < 1$  is such that the last expression becomes true, namely there exists an  $M > 0$  and

$$\left| \int_{\frac{2}{z}}^{\infty} k(x, z) dx - 96z^5 \right| \leq M |z^7|$$

for  $0 < z < \epsilon$ .

Thus we have

$$0 < \mu(z) < \frac{30mc}{d} \frac{z^6}{(2z^2 + 4z + 1)^2} (96z^5 + \mathcal{O}(z^6))^{-1},$$

which formally shows that  $\mu \rightarrow 0^+$  as  $z \rightarrow 0^+$  in the non relativistic limit.  $\square$

The ultrarelativistic case is established in the following.

**Theorem 2.**  $\mu(z) \rightarrow 0^+$  as  $z \rightarrow +\infty$ .

*Proof.* We realize that  $\frac{2}{z} \rightarrow 0$ , therefore

$$\begin{aligned} \lim_{z \rightarrow \infty} \mathcal{I}(z) &= \left( \lim_{z \rightarrow \infty} z^5 \right) \int_0^{\infty} e^{-x^2} x^2 (x^2 + 3x + 3) dx \\ &= 48 \lim_{z \rightarrow \infty} z^5. \end{aligned}$$

Thus

$$\lim_{z \rightarrow \infty} \mu(z) = \frac{5}{8} \frac{mc}{d} \lim_{z \rightarrow \infty} \frac{z}{(2z^2 + 4z + 1)^2} = 0.$$

as is claimed  $\square$

Theorem 1 and 2 claim that the bulk viscosity becomes negligible in the limits. This implies that, whatever effect it has on the dynamics of the fluids, it should only be relevant in a finite interval of  $z$ . In order to assess the possible effect of bulk viscosity dissipation and provide an example of an experiment that may lead to the measure of such effect, we carry out a linear analysis of a free relativistic gas to first order in statistical density fluctuations.

## 5. Modification to the Brillouin peaks due to bulk viscosity

The transport equation for the relativistic gas can be readily established by multiplying Eq. (2) by the collisional invariants and integrating in velocity space. The procedure is the standard one, and leads to two conservation equations

$$N_{,\nu}^{\nu} = 0, \quad T_{,\nu}^{\mu\nu} = 0 \quad (24)$$

where the particle flux is given by

$$N^{\nu} = \int f(v^{\nu}) v^{\nu} dv^* \quad (25)$$

and the energy momentum tensor is

$$T^{\mu\nu} = m \int f(v^{\nu}) v^{\mu} v^{\nu} dv^*. \quad (26)$$

The relation of such moments with the state variables is given, in Eckart's frame and using the (2+1) decomposition by  $N^{\nu} = nu^{\nu}$  and  $T^{\mu\nu} = n\epsilon u^{\mu} u^{\nu} / c^2 + ph^{\mu\nu} + \pi^{\mu\nu} + q^{\mu} u^{\nu} / c^2 + u^{\mu} q^{\nu} / c^2$ . Such expressions are then introduced in the balance equations (Eq. (24, 25)) and the state variables

are assumed to be given by an equilibrium value plus a small fluctuation:  $X = X_0 + \delta X$ . The resulting system of equations, to first order in fluctuations ( $\delta n$ ,  $\delta u^\mu$  and  $\delta T$ ) can be then transformed to Fourier-Laplace space, in which the corresponding dispersion relation is given by

$$s^3 + a_1 q^2 s^2 + (a_2 q^2 + a_3) q^2 s + a_4 q^4 = 0 \quad (27)$$

where

$$\begin{aligned} a_1 &= -\frac{1}{\rho} \left( A - \frac{\rho}{p} k_p(z) L_T + \frac{1}{c^2} (L_n + k_p(z) L_T) \right), \\ a_2 &= -\frac{k_p(z)}{\rho} A L_T, \\ a_3 &= \frac{p}{\rho} (1 + k_p(z)), \\ a_4 &= \frac{k_p(z)}{\rho} (L_T - L_n), \end{aligned}$$

here bulk viscosity enters in Eq. (27) through the relation  $A = 4\eta/3 + \mu$  where  $\eta$  is the shear viscosity, and  $L_T$  and  $L_n$  are the transport coefficients appearing in the relativistic heat flux constitutive equation [3, 7]. Equation (27) has the same structure as the dispersion relation in the non-relativistic case which can be analyzed using Mountain's method [8]. Following such approximation, one can identify a purely decaying mode, corresponding to a real root given by  $s_1 = -a_4 q^2 / a_3$ . The remaining two roots correspond to a conjugate pair

$$s_{2,3} = -\frac{1}{2} \left( a_1 + \frac{a_4}{a_3} \right) \pm i q \sqrt{a_3} \quad (28)$$

which leads to decaying, oscillating modes. Thus, the corrections due to the relativistic nature of the molecular dynamics of the disks could be measured in a light scattering experiment where a Rayleigh-Brillouin spectrum can be obtained. In particular, since the focus of this work is the effect of the bulk viscosity, we are only concerned with the Brillouin doublet width, which is given by the real part of the complex roots (Eq. (28)). The width of the central peak as well as the location of the lateral ones is not affected by the presence of  $\mu$ . Let's call

$$W(\mu(z)) = -\frac{1}{2} \left( a_1 + \frac{a_4}{a_3} \right), \quad (29)$$

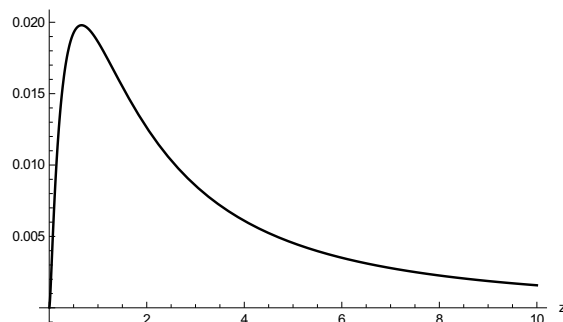
the width of the Brillouin peaks. Also, in order to understand better the effect of  $\mu$ , in Fig. 2 we show the ratio  $(W(\mu) - W(\mu = 0)) / W(\mu = 0)$  as a function of  $z$ . Clearly  $W(\mu \neq 0)$  and  $W(\mu = 0)$  are equal at  $z = 0$ , the ratio reaches a maximum for an intermediate value of  $z$  and finally tends to zero for large values of the temperature.

## 6. Discussion and final remarks

The integral in Eq. (1) that results in a complete identification of the dependence of  $\mu$  with the temperature in the two-dimensional relativistic system is far of being simple. In this work we showed that in both the non-relativistic and ultrarelativistic limits the bulk viscosity of a 2D relativistic system vanishes and the formal proof was presented.

Dissipation in single component fluids in the absence of external forces is composed of two effects: viscous and thermal. While thermal dissipation in relativistic systems modifies in a somewhat dramatical fashion the structure of the transport equations, the viscous tensor effects remain the same in form however the coefficients are significantly altered. In particular, the Rayleigh-Brillouin spectrum is modified by the presence of a non-vanishing bulk viscosity in





**Figure 2.** The ratio of the Brillouin widths  $(W(\mu) - W(\mu = 0)) / W(\mu = 0)$  as a function of  $z$ .

the relativistic scenario. The dynamics of density fluctuations clearly depend on all dissipative contributions. In general, the shape of the Brillouin peaks is given in terms of all the transport coefficients (see Eq. (29)) but the presence of a non zero bulk coefficient in the intermediate temperature regime leads to a slight modification of the spectrum, in particular the width of the lateral peaks is altered, as is shown in Fig. 2 and Eq. (29). The analysis of the extreme limits shown in this paper for all the transport coefficients in the (2+1) case is important in the context of the development of new two-dimensional materials as graphene and will be addressed elsewhere.

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