Research Article

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Gaussons: optical solitons with log-law nonlinearity by Laplace–Adomian decomposition method

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Abstract: This paper presents optical Gaussons by the aid of the Laplace–Adomian decomposition scheme. The numerical simulations are presented both in the presence and in the absence of the detuning term. The error analyses of the scheme are also displayed.

Keywords: Gaussons, log-law nonlinearity, nonlinear Schrödinger equation, Adomian decomposition method, Laplace transform

1 Introduction

The study of optical solitons with log-law nonlinearity, also known as optical Gaussons, has gained popularity during the last decade. Several analytical results have been reported [1–9]. In this context, soliton perturbation theory, quasi-stationary optical Gaussons, and birefringent fibers with dense wavelength division multiplexing (DWDM) technology have been addressed. Recently, shifting gears, the interest on the nonlinear Schrödinger's equation (NLSE) with log-law nonlinearity, has emerged in the direction of numerical studies [10]. The travelling wave solution technology for finding the partial differential equations (PDEs) was developed by various studies [11–18]. This paper addresses optical Gaussons using the Laplace–Adomian decomposition method (LADM). This is a modified version of the popular Adomian decomposition method (ADM) that has gained extreme popularity during the last decade or so. ADM has been successfully implemented to a wide variety of nonlinear evolution equations, and several impressive numerical results have been reported. In this work, the LADM scheme will be first derived and discussed in detail and subsequently implemented to NLSE with log-law nonlinearity. The model will be studied both in the presence and in the absence of the detuning term. The numerical simulations all appear with their respective error analyses, and these are all depicted in respective tables and figures.

2 Governing equation and optical Gaussons

2.1 The model

To study the behavior of Gaussons, we consider the dimensionless form of the NLSE with log-law nonlinearity considered in ref. [19–21] and given by

$$iu_t + au_{xx} + bu \ln |u|^2 = f(x)u, \tag{1}$$

where $u = u(x, t)$ is the complex amplitude of the wave and x and t are spatial and temporal coordinates, respectively. In the optical fiber contexts, the coefficient a is the group velocity dispersion and b is the coefficient of log-law nonlinearity. On the right-hand side of equation (1), $f(x)$ represents the spatially dependent detuning term [22,23].

Although the usual norm is to study solitons in a Kerr nonlinear medium, there are some advantages to study log-law nonlinearity over Kerr law nonlinearity. One immediate advantage is that solitons from Kerr law

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nonlinearity produce radiation that is not present for NLSE in a log-law medium [24].

2.2 Optical Gaussons

2.2.1 Case $f(x) = 0$

In this case, the one-soliton solution to (1), also known as Gaussons, is given by [4,25]

$$u(x,t) = Ae^{-B^2(x-vt)^2}e^{i\phi(x,t)}, \tag{2}$$

where A is the amplitude of the Gaussons and B is related to the coefficients of NLSE (1) by means of

$$B = \sqrt{\frac{b}{2a}}. \tag{3}$$

Now, choosing the phase as

$$\phi(x,t) = -\kappa x + \omega t + \theta \tag{4}$$

where κ represents the frequency of the Gaussons, while θ and ω are the phase and the wavenumber of the soliton, respectively.

The velocity of the Gausson is obtained from the frequency and coefficients of the model (1) and is given by

$$v = -2a\kappa \tag{5}$$

and the wave number is

$$\omega = 2b\ln(A) - a\kappa^2 - b. \tag{6}$$

Observing equation (3), the constraint condition that guarantees the existence of Gaussons is given as follows:

$$ab > 0. \tag{7}$$

Finally, in the present case, the optical Gaussons solution of the NLSE with log-law nonlinearity is given by

$$u(x,t) = Ae^{-B^2(x-vt)^2}e^{i[-\kappa x+\omega t+\theta]}. \tag{8}$$

2.2.2 Case $f(x) = n$

In this case, the starting hypothesis for log-law nonlinearity is given by [4,22,26]

$$u(x,t) = Ae^{(Ct-D^2x^2)}e^{i[-\kappa x+\omega t+\theta]}. \tag{9}$$

where A is the amplitude of the Gaussons and the constants C and D are related to the dynamics of the model by [22]

$$C = \frac{b-\kappa+n}{2b} \quad D^2 = \frac{b}{2a}. \tag{10}$$

From the phase component, the wavenumber ω is given by

$$\omega = 2b\ln(A) - a\kappa^2 - b. \tag{11}$$

Finally, equation (10) prompts the constraint condition as follows:

$$ab > 0. \tag{12}$$

3 Analysis of the methodology

The Adomian method combined with the Laplace transform LADM is a decomposition method that gives us solutions for nonlinear differential equations in terms of a convergent series [27]. The LADM was established and used for the first time by Khuri [28] for solving differential equations.

We consider the general form of nonlinear partial differential equations with the initial condition in the following equation:

$$\begin{gathered} L_t u(x,t) + Ru(x,t) + Nu(x,t) = h(x,t), \\ u(x,0) = g(x) \end{gathered} \tag{13}$$

where $L_t u = u_t$, R is a linear operator that includes partial derivatives with respect to x, N is a nonlinear operator, and the function h is a source term, which must be independent of u.

Solving for $L_t u(x,t)$ and applying Laplace transform on both sides of equation (13), we obtain

$$\mathcal{L}\{L_t u(x,t)\} = \mathcal{L}\{h(x,t) - Ru(x,t) - Nu(x,t)\}. \tag{14}$$

Using the known property of the Laplace transform with respect to the first derivative, that is, $\mathcal{L}\{u_t(x,t)\} = s\mathcal{L}\{u(x,t)\} - u(x,0)$, the expression (14) is equivalent to

$$su(x,s) - u(x,0) = \mathcal{L}\{h(x,t) - Ru(x,t) - Nu(x,t)\} \tag{15}$$

In the homogeneous case, because $\mathcal{L}$ is a linear operator, we have

$$u(x,s)=\frac{g(x)}{s}-\frac{1}{s}\mathcal{L}\{Ru(x,t)+Nu(x,t)\}. \quad (16)$$

Now, considering the initial condition given in equation (13), $u(x, t)$ are obtained easily by applying the inverse Laplace transform $\mathcal{L}^{-1}$ on both sides of equation (16)

$$u(x,t)=g(x)-\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\{Ru(x,t)+Nu(x,t)\}\right]. \quad (17)$$

The ADM decomposes $u(x, t)$ as a series with components $u_n(x, t)$ and $N(u)$ as a series with components $A_n(u_0, u_1,...,u_n)$, namely,

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t) \quad (18)$$

$$Nu(x,t)=\sum_{n=0}^{\infty}A_n(u_0,u_1,\ldots,u_n). \quad (19)$$

In equation (19), $\{A_n\}_{n=0}^{\infty}$ is the so-called Adomian polynomials sequence established in ref. [29], which are calculated recursively by

$$A_n(u_0,\ldots,u_n)=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=1}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0}, \quad n=0,1,2,\ldots \quad (20)$$

Substituting (18) and (19) into equation (17) gives rise to

$$\sum_{n=0}^{\infty}u_n(x,t)=g(x)-\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left\{R\sum_{n=0}^{\infty}u_n(x,t)+\sum_{n=0}^{\infty}A_n(u_0,u_1,\ldots,u_n)\right\}\right]. \quad (21)$$

From equation (21), each of the components of the series (18) is obtained by means of the recursion scheme, for each $n = 0, 1, 2,...$

$$\begin{cases}u_0(x,t)=g(x),\\ u_{n+1}(x,t)=-\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\{Ru_n(x,t)+A_n(u_0,\ldots,u_n)\}\right].\end{cases} \quad (22)$$

The convergence of this series has been studied in ref. [30]. Finally, for numerical purposes, the N-term approximate

$$u_N(x,t)=\sum_{n=0}^{N-1}u_n(x,t),\quad N\geq 1, \quad (23)$$

can be used to approximate the exact solution. The exact solution is obtained by adding all the terms, that is,

$$u(x,t)=\lim_{N\to\infty}u_N(x,t). \quad (24)$$

Now we will implement the algorithm provided by LADM for the solution of NLSE with log-law nonlinearity.

3.1 Gaussons solution of the NLSE with log-law nonlinearity through LADM

In this section, the Laplace transform decomposition algorithm given by equation (22) is applied to find the solution to the following NLSE with log-law nonlinearity (1):

$$\begin{aligned}u_t(x,t)&=iau_{xx}(x,t)-if(x)u(x,t)\\&\quad+ibu(x,t)\ln|u(x,t)|^2.\end{aligned} \quad (25)$$

Comparing (25) and (13), we identify the operators L_t and R by

$$L_t u(x,t)=u_t(x,t),\quad Ru=i(a\partial_x^2-f(x))u(x,t), \quad (26)$$

whereas that the nonlinear term turns out to be

$$Nu=ibu(x,t)\ln|u(x,t)|^2=ib\sum_{n=0}^{\infty}A_n(u_0,\ldots,u_n), \quad (27)$$

Using (20), the first Adomian's Polynomials of $N(u)$ are as given as follows:

$$\begin{aligned}A_0&=u_0\ln|u_0\bar{u}_0|,\\ A_1&=\frac{u_0u_1\bar{u}_0'}{\bar{u}_0}+u_1\ln|u_0\bar{u}_0|+u_1,\\ A_2&=\frac{u_0u_1^2}{2\bar{u}_0}\bar{u}_0''+\frac{u_1^2}{\bar{u}_0}\bar{u}_0'-\frac{u_0u_1^2}{2\bar{u}_0^2}\bar{u}_0^{2'}+\frac{u_0u_2}{\bar{u}_0}\bar{u}_0'\\&\quad+u_2\ln|u_0\bar{u}_0|+\frac{u_1^2}{2u_0}+u_2,\end{aligned}$$

$$
\begin{aligned}
A_3 = {} & \frac{u_0u_1^3}{6\bar{u}_0}\bar{u}_0''' + \frac{u_1^3}{2\bar{u}_0}\bar{u}_0'' + \frac{u_0u_2u_1}{\bar{u}_0}\bar{u}_0'' + \frac{u_0u_1^3}{3\bar{u}_0^3}\bar{u}_0^{3'} \\
& - \frac{u_1^3}{2\bar{u}_0^2}\bar{u}_0^{2'} + \frac{2u_1u_2}{\bar{u}_0}\bar{u}_0' - \frac{u_0u_1u_2}{\bar{u}_0^2}\bar{u}_0^{2'} + \frac{u_0u_3}{\bar{u}_0}\bar{u}_0' \\
& - \frac{u_0u_1^3}{2\bar{u}_0^2}\bar{u}_0'\bar{u}_0'' + u_3\ln|u_0\bar{u}_0| - \frac{u_1^3}{6u_0^2} + \frac{u_1u_2}{u_0} + u_3, \\
A_4 = {} & \frac{u_0u_1^4}{24\bar{u}_0}\bar{u}_0'''' + \frac{u_1^4}{6\bar{u}_0}\bar{u}_0''' + \frac{u_0u_2u_1^2}{2\bar{u}_0}\bar{u}_0''' - \frac{u_0u_1^4}{8\bar{u}_0^2}\bar{u}_0''^2 \\
& + \frac{3u_2u_1^2}{2\bar{u}_0}\bar{u}_0'' + \frac{u_0u_1u_3}{\bar{u}_0}\bar{u}_0'' + \frac{u_0u_2^2}{2\bar{u}_0}\bar{u}_0'' + \frac{u_1^4}{3\bar{u}_0^3}\bar{u}_0'^3 \\
& - \frac{u_0u_1^4}{4\bar{u}_0^4}\bar{u}_0'^4 + \frac{u_0u_2u_1^2}{\bar{u}_0^3}\bar{u}_0'^3 - \frac{3u_2u_1^2}{2\bar{u}_0^2}\bar{u}_0'^2 + \frac{2u_3u_1}{\bar{u}_0}\bar{u}_0' \\
& - \frac{u_0u_1u_3}{\bar{u}_0^2}\bar{u}_0'^2 + \frac{u_2^2}{\bar{u}_0}\bar{u}_0' + \frac{u_0u_4}{\bar{u}_0}\bar{u}_0' - \frac{u_0u_2^2}{2\bar{u}_0^2}\bar{u}_0'^2 \\
& - \frac{u_0u_1^4}{6\bar{u}_0^2}\bar{u}_0'\bar{u}_0''' + \frac{u_0u_1^4}{2\bar{u}_0^3}\bar{u}_0'^2\bar{u}_0'' - \frac{u_1^4}{2\bar{u}_0^2}\bar{u}_0'\bar{u}_0'' \\
& - \frac{3u_0u_2u_1^2}{2\bar{u}_0^2}\bar{u}_0'\bar{u}_0'' \\
& + u_4\ln|u_0\bar{u}_0| + \frac{u_1^4}{12u_0^3} - \frac{u_2u_1^2}{2u_0^2} + \frac{u_1u_3}{u_0} + \frac{u_2^2}{2u_0} + u_4, \\
& \vdots
\end{aligned}
$$

From equation (22), we achieve the series solutions as follows:

$$
\begin{aligned}
u_0(x,t) &= g(x), \\
u_1(x,t) &= \mathcal{L}^{-1}\Big[\frac{i}{s}\mathcal{L}\{au_{0,xx}(x,t) - f(x)u_0(x,t) \\
&\quad + bA_0(u_0)\}\Big], \\
u_2(x,t) &= \mathcal{L}^{-1}\Big[\frac{i}{s}\mathcal{L}\{au_{1,xx}(x,t) - f(x)u_1(x,t) \\
&\quad + bA_1(u_0,u_1)\}\Big], \\
&\ \ \vdots \\
u_n(x,t) &= \mathcal{L}^{-1}\Big[\frac{i}{s}\mathcal{L}\{au_{n-1,xx}(x,t) - f(x)u_{n-1}(x,t) \\
&\quad + bA_{n-1}(u_0,\ldots,u_{n-1})\}\Big].
\end{aligned}
$$

In Section 4, we illustrate the LADM for solving the NLSE with log-law nonlinearity (1) for different special cases.

4 Numerical simulations

In this section, some examples are provided to show the reliability and the efficiency of the proposed method in solving nonlinear differential equations used in the modeling of dynamics in quantum optics of the type (1). Our computations are performed by MATHEMATICA software.

4.1 Gaussons without detuning term

Table 1 and Figures 1–3.

Table 1: Gaussons without detuning term

Cases	a	b	A	B	κ	v	ω	N	|Max error|
1	0.5	1.0	2.0	1.00	0.5	−0.5	0.26	12	1.0×10^{-10}
2	0.3	2.0	2.5	1.82	1.0	−0.6	1.36	12	1.5×10^{-10}
3	0.2	3.0	3.0	2.73	1.0	−0.4	3.39	12	1.5×10^{-9}

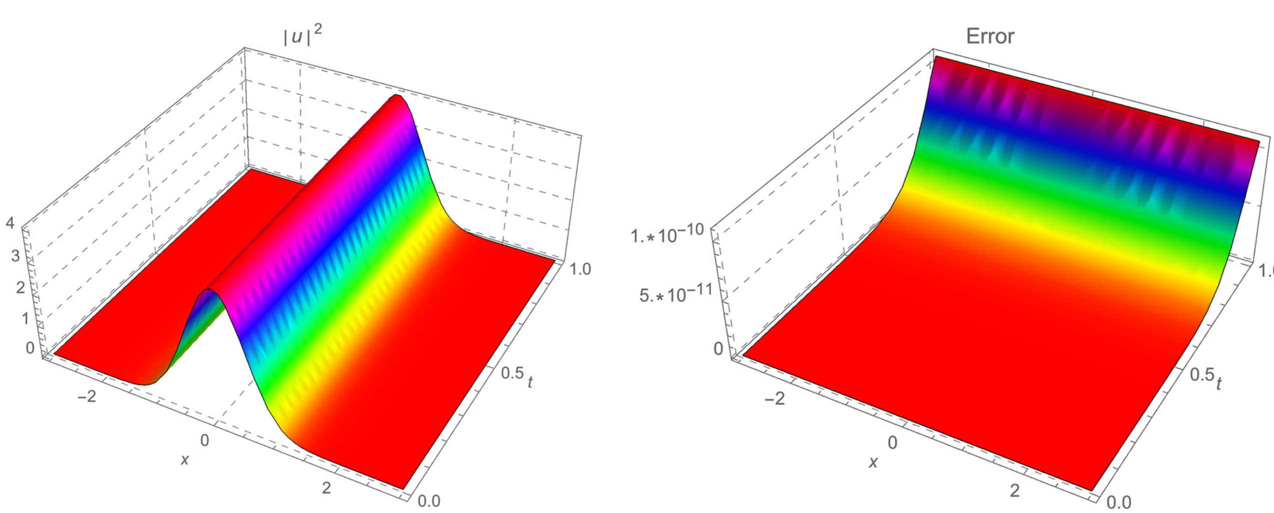

Figure 1: Gausson without detuning term: case 1: numerically computed profile (left) and absolute error (right).

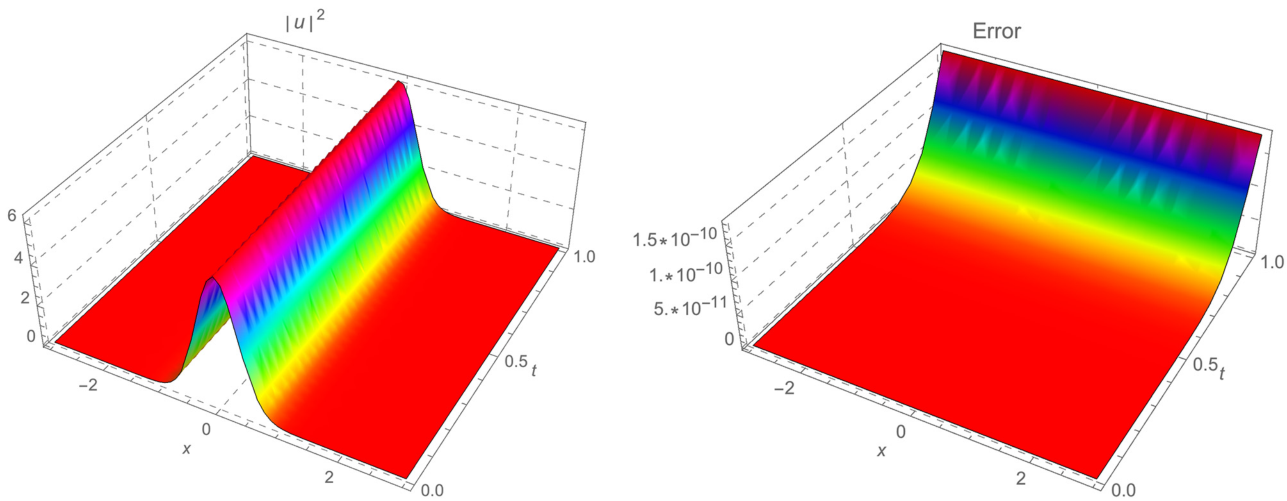

Figure 2: Gausson without detuning term: case 2: numerically computed profile (left) and absolute error (right).

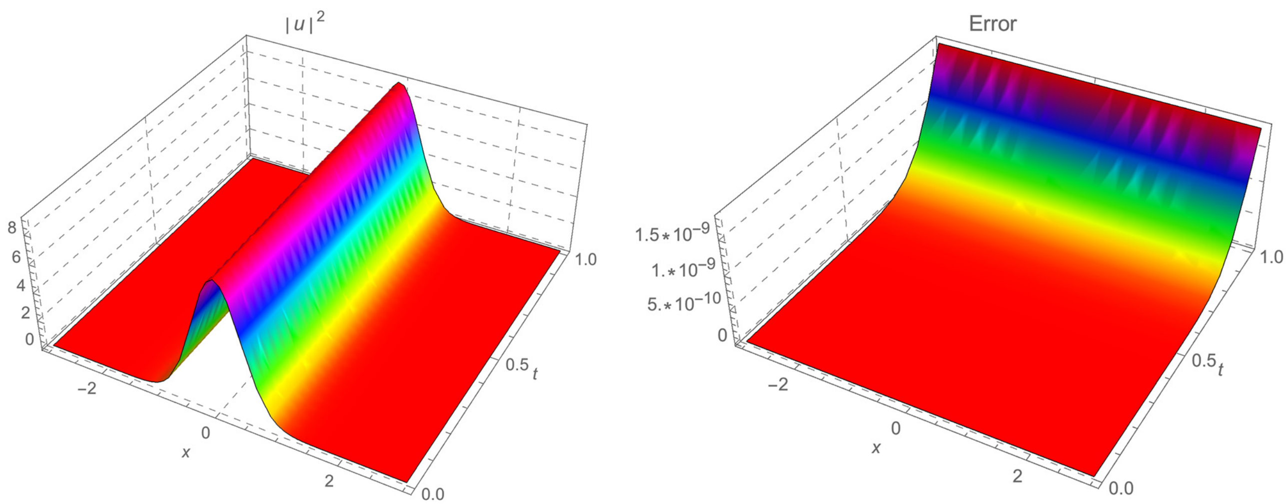

Figure 3: Gausson without detuning term: case 3: numerically computed profile (left) and absolute error (right).

Table 2: Gaussons with detuning term

Cases	a	b	A	C	D	κ	ω	$f(x)$	N	|Max error|
1	0.5	1.0	2.5	0.50	1.0	1.00	0.33	1	12	4.0×10^{-10}
2	0.3	0.2	3.0	4.25	0.57	0.5	0.16	2	12	6.0×10^{-10}
3	0.2	0.3	4.0	5.00	0.86	0.3	0.51	3	12	2.0×10^{-9}

4.2 Gaussons with detuning term

Table 2 and Figures 4–6.

5 Conclusions

This paper successfully studied optical Gaussons that emerged from NLSE with log-law nonlinearity by the aid of LADM. The results of this paper are being reported for the first time. The error analysis proved that the results appear with grand success and are truly impressive. The results of this paper thus stand on a strong footing to move further along. Later, this scheme will be implemented to address the model further along. They are the application of the scheme to study Gaussons in birefringent fibers, DWDM systems, as well as optical couplers, photonic crystal fiber (PCF), metamaterials,

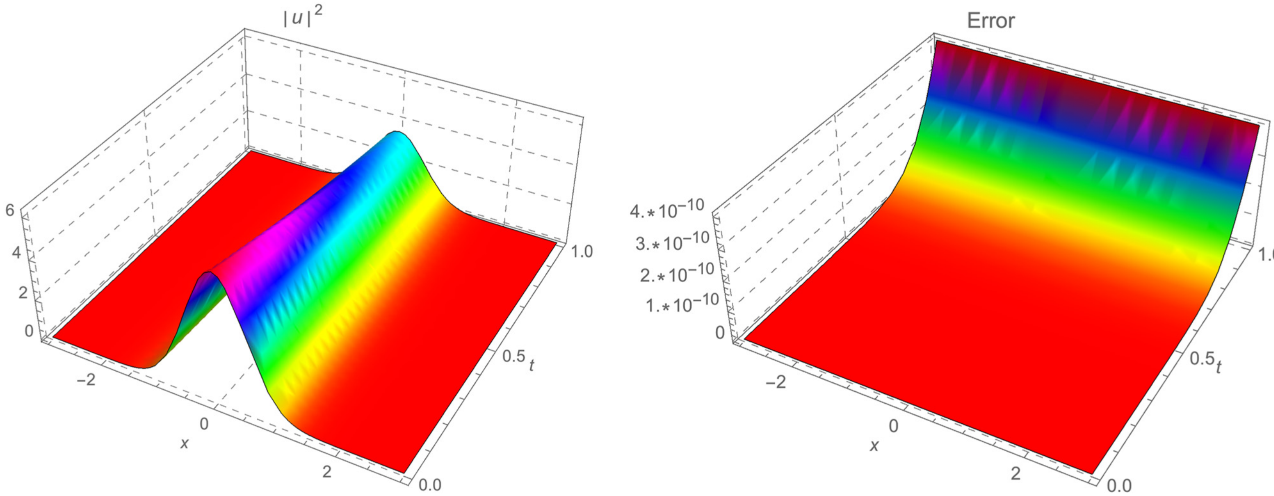

Figure 4: Gausson with detuning term: case 1: numerically computed profile (left) and absolute error (right).

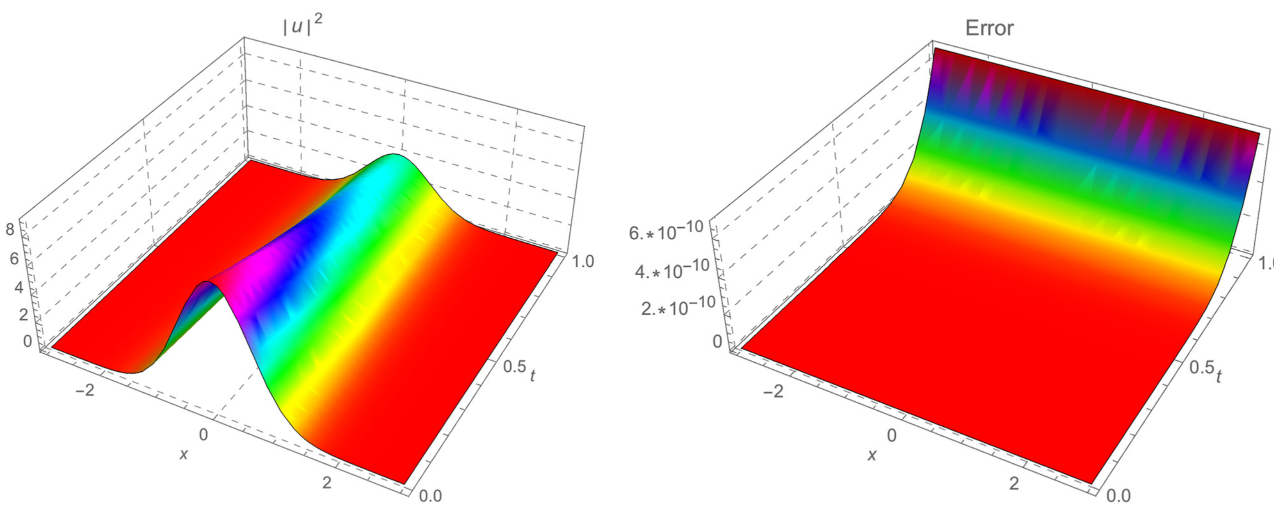

Figure 5: Gausson with detuning term: case 2: numerically computed profile (left) and absolute error (right).

and other variety of optoelectronic devices. Moreover, additional numerical schemes will be implemented to address the model in such optoelectronic devices; one such scheme is the variational iteration method. Such studies are underway. This is just the tip of the iceberg.

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