RESEARCH Revista Mexicana de Física **66** (3) 291–296 MAY-JUNE 2020

Highly dispersive optical solitons having Kerr law of refractive index with Laplace-Adomian decomposition

O. González-Gaxiola[a,*], Anjan Biswas,[b,c,d,e] and Ali Saleh Alshomrani[c]
[a]*Departamento de Matemáticas Aplicadas y Sistemas,*
Universidad Autónoma Metropolitana-Cuajimalpa. Vasco de Quiroga 4871, 05348 Mexico City, Mexico.
[*]*ogonzalez@correo.cua.uam.mx*
[b]*Department of Physics, Chemistry and Mathematics,*
Alabama A& M University, Normal, AL 35762-7500, USA.
[c]*Department of Mathematics, King Abdulaziz University, Jeddah-21589, Saudi Arabia.*
[d]*Department of Applied Mathematics, National Research Nuclear University,*
31 Kashirskoe Shosse, Moscow-115409, Russian Federation.
[e]*Department of Mathematics and Statistics, Tshwane University of Technology, Pretoria-0008, South Africa.*

Received 5 November 2019; accepted 16 December 2019

This paper studies highly dispersive optical solitons, having Kerr law of refractive index, numerically. The adopted scheme is Laplace-Adomian decomposition method. Bright soliton solutions are displayed along with their respective error analysis.

Keywords: Nonlinear Schrödinger equation; bright solitons; higher-order dispersion; Laplace-adomian decomposition method.

PACS: 02.30.Jr; 02.30.Uu; 02.60.Cb; 44.05.+e DOI: https://doi.org/10.31349/RevMexFis.66.291

1. Introduction

The concept of highly dispersive optical solitons emerged during 2019 as an extension and/or generalization to cubic-quartic solitons. Analytical results are abundant and stem from this concept, and these have been recovered after implementing of algorithms. These include extended trial function method, F-expansion scheme, Jacobi's elliptic function expansion, exp-expansion and others [1–5]. The conservation laws for such solitons have also been reported [3]. Therefore, it is now time to turn the page and explore this topic from a numerical perspective. This paper, therefore. addressed highly dispersive optical solitons, having Kerr law of refractive index, by the aid of the Laplace-Adomian decomposition scheme. The focus on this paper will be on bright optical solitons. The scheme is first explicitly elaborated and subsequently implemented into the model equation successfully. The details are sketched in the rest of the paper.

2. Governing equation

The dimensionless form of NLSE with Kerr law nonlinearity in presence of dispersion terms of all orders is [6]:

$$\begin{aligned} &iq_t + ia_1 q_x + a_2 q_{xx} + ia_3 q_{xxx} + a_4 q_{xxxx} \\ &\quad + ia_5 q_{xxxxx} + a_6 q_{xxxxxx} + b|q|^2 q = 0, \end{aligned} \tag{1}$$

where $q = q(x,t)$ is a complex-valued function of x (space) and t (time) and $i = \sqrt{-1}$. The first term represents linear temporal evolution. The next six terms are dispersion terms that make the solitons highly dispersive. These are given by the coefficients of a_k for $1 \leq k \leq 6$, which are intermodal dispersion (IMD), group velocity dispersion (GVD), third-order dispersion (3OD), fourth-order dispersion (4OD), fifth-order dispersion (5OD) and sixth-order dispersion (6OD) respectively. Finally, b indicates the coefficient of self-phase modulation based on cubic or Kerr nonlinearity.

2.1. Bright solitons

The bright 1-soliton solution to (1) was recently found by the authors in [6, 7] using the semi-variational principle and is given by

$$\begin{aligned} q(x,t) = A\Big(&\operatorname{sech}[B(x-\nu t)] \\ &+ \operatorname{sech}^3[B(x-\nu t)]\Big) e^{i[-\kappa x + \omega t + \theta_0]}. \end{aligned} \tag{2}$$

In Eq. (2), ν is the soliton velocity, ω is the angular velocity, κ is the soliton frequency, and θ_0 is the phase center.

In [7], the amplitude A of the 1-soliton was calculated as:

$$A = \left[-\frac{182.946 a_6 B^6 - 16.874 P_3 B^4 + 11.847 P_2 B^2}{26.118 b} \right]^{\frac{1}{2}}, \tag{3}$$

where:

$$\begin{aligned} P_1 = &-\omega + a_1\kappa - a_2\kappa^2 - a_3\kappa^3 \\ &+ a_4\kappa^4 + a_5\kappa^5 - a_6\kappa^6 \end{aligned} \tag{4}$$

$$P_2 = a_2 + 3a_3\kappa - 6a_4\kappa^2 - 10a_5\kappa^3 + 15a_6\kappa^4 \tag{5}$$

$$P_3 = a_4 + 5a_5\kappa - 15a_6\kappa^2. \tag{6}$$

Besides, the inverse width B of the 1-soliton is a real root of the equation:

$$670.813a_6B^6 - 59.059P_3B^4 + 35.541P_2B^2 - 9933P_1 = 0. \quad (7)$$

The velocity ν is given by

$$\nu = a_1 - 2a_2\kappa - 3a_3\kappa^2 + 5a_5\kappa^4 - 6a_6\kappa^5. \quad (8)$$

Finally, there are the following two relationships between the soliton frequency κ and some of the coefficients of Eq. (1), these are given by [8]

$$a_3 - 4a_4\kappa - 10a_5\kappa^2 + 20a_6\kappa^3 = 0 \quad \text{and} \quad a_5 - 6a_6\kappa = 0.$$

3. Method applied

In this section, we will describe the basic theory and algorithm of the Laplace-Adomian decomposition method (LADM), used to solve nonlinear partial differential equations, and that was first proposed in [9, 10].

Let us look for soliton solutions of Eq. (1) in the form $q(x,t) = u(x,t) + iv(x,t)$. Then we can decompose the Eq. (1) in its real and imaginary parts, respectively as

$$u_t = -a_1u_x - a_2v_{xx} - a_3u_{xxx} - a_4v_{xxxx} - a_5u_{xxxxx} - a_6v_{xxxxxx} - bv(u^2+v^2) \quad (9)$$

$$v_t = -a_1v_x + a_2u_{xx} - a_3v_{xxx} + a_4u_{xxxx} - a_5v_{xxxxx} + a_6u_{xxxxxx} + bu(u^2+v^2) \quad (10)$$

To give analytical approximate solutions for Eq. (1) using LADM, we first rewrite the Eqs. (9) and (10) in the following operator form

$$D_tu = -a_1D_x^1u - a_2D_x^2v - a_3D_x^3u - a_4D_x^4v - a_5D_x^5u - a_6D_x^6v + N_1(u,v) \quad (11)$$

$$D_tv = -a_1D_x^1v + a_2D_x^2u - a_3D_x^3v + a_4D_x^4u - a_5D_x^5v + a_6D_x^6u + N_2(u,v) \quad (12)$$

with initial conditions $u(x,0) = \Re e(q(x,0))$ and $v(x,0) = \Im m(q(x,0))$.

In the equations system (11)-(12), the operator D_t denotes derivative with respect to t, whereas that D_x^j is the $j-$th order linear differential operator $\partial^j/\partial x^j$, and N_k represents nonlinear differential operators for $k = 1, 2$.

The method consists of first applying the Laplace transform $\mathcal{L}$ to both sides of equations in system (11)-(12) and then, by using initial conditions, we have

$$u(x,s) = \frac{u(x,0)}{s} + \frac{1}{s}\mathcal{L}\{-a_1D_x^1u - a_2D_x^2v - a_3D_x^3u - a_4D_x^4v - a_5D_x^5u - a_6D_x^6v + N_1(u,v)\} \quad (13)$$

$$v(x,s) = \frac{v(x,0)}{s} + \frac{1}{s}\mathcal{L}\{-a_1D_x^1v + a_2D_x^2u - a_3D_x^3v + a_4D_x^4u - a_5D_x^5v + a_6D_x^6u + N_2(u,v)\} \quad (14)$$

Thus, by applying the inverse Laplace transform $\mathcal{L}^{-1}$, we obtain

$$u(x,t) = u(x,0) + \mathcal{L}^{-1}\Big[\frac{1}{s}\mathcal{L}\{-a_1D_x^1u - a_2D_x^2v - a_3D_x^3u - a_4D_x^4v - a_5D_x^5u - a_6D_x^6v + N_1(u,v)\}\Big] \quad (15)$$

$$v(x,t) = v(x,0) + \mathcal{L}^{-1}\Big[\frac{1}{s}\mathcal{L}\{-a_1D_x^1v + a_2D_x^2u - a_3D_x^3v + a_4D_x^4u - a_5D_x^5v + a_6D_x^6u + N_2(u,v)\}\Big] \quad (16)$$

According to the standard Adomian decomposition method, the solutions u and v can be expressed in an infinite series as follows

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \qquad v(x,t) = \sum_{n=0}^{\infty} v_n(x,t) \quad (17)$$

Also, the nonlinear terms can be written as

$$N_1(u,v) = -b(vu^2+v^3) = -b\sum_{n=0}^{\infty} A_n(u_0,u_1,\ldots,u_n;v_0,v_1,\ldots,v_n) \quad (18)$$

and

$$N_2(u,v) = b(uv^2+u^3) = b\sum_{n=0}^{\infty} B_n(u_0,u_1,\ldots,u_n;v_0,v_1,\ldots,v_n) \quad (19)$$

where A_n and B_n are the Adomian?s polynomials [11, 12], which are defined by

$$A_n(u_0,\ldots,u_n;v_0,\ldots,v_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n} \times \Big[N_1\Big(\sum_{i=1}^{\infty}\lambda^iu_i; \sum_{i=1}^{\infty}\lambda^iv_i\Big)\Big]_{\lambda=0}, \quad n = 0,1,2,\ldots \quad (20)$$

$$B_n(u_0,\ldots,u_n;v_0,\ldots,v_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n} \times \Big[N_2\Big(\sum_{i=1}^{\infty}\lambda^iu_i; \sum_{i=1}^{\infty}\lambda^iv_i\Big)\Big]_{\lambda=0}, \quad n = 0,1,2,\ldots \quad (21)$$

On making the substitution of Eqs. (17) and (18) into Eqs. (15) and (16), we can arrive at

$$\sum_{n=0}^{\infty} u_n = u(x,0) + \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\{-(a_1 D_x^1 + a_3 D_x^3 + a_3 D_x^5)\sum_{n=0}^{\infty} u_n - (a_2 D_x^2 + a_4 D_x^4 + a_6 D_x^6)\sum_{n=0}^{\infty} v_n - b\sum_{n=0}^{\infty} A_n\}\right] \quad (22)$$

$$\sum_{n=0}^{\infty} v_n = v(x,0) + \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\{-(a_1 D_x^1 + a_3 D_x^3 + a_3 D_x^5)\sum_{n=0}^{\infty} v_n + (a_2 D_x^2 + a_4 D_x^4 + a_6 D_x^6)\sum_{n=0}^{\infty} u_n + b\sum_{n=0}^{\infty} B_n\}\right] \quad (23)$$

Now, according to LADM, The following recursive schemes can be constructed

$$\begin{cases} u_0(x,t) = \Re e(q(x,0)), \\ u_{n+1}(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\{-(a_1 D_x^1 + a_3 D_x^3 + a_3 D_x^5)u_n - (a_2 D_x^2 + a_4 D_x^4 + a_6 D_x^6)v_n - bA_n\}\right], \quad n \geq 0, \end{cases} \quad (24)$$

$$\begin{cases} v_0(x,t) = \Im m(q(x,0)), \\ u_{n+1}(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\{-(a_1 D_x^1 + a_3 D_x^3 + a_3 D_x^5)v_n + (a_2 D_x^2 + a_4 D_x^4 + a_6 D_x^6)u_n + bB_n\}\right], \quad n \geq 0. \end{cases} \quad (25)$$

Being $N_1(u,v) = vu^2 + v^3$ and $N_2(u,v) = uv^2 + u^3$, using the formulas (20) and (21) some terms of Adomian's polynomials A_n and B_n are given by

$$\begin{aligned} A_0 &= u_0^2 v_0 + v_0^3, \\ A_1 &= 2u_0 u_1 v_0 + u_0^2 v_1 + 3v_0^2 v_1, \\ A_2 &= 2u_0 u_2 v_0 + u_1^2 v_0 + 2u_0 u_1 v_1 + u_0^2 v_2 \\ &\quad + 3v_0^2 v_2 + 3v_0 v_1^2, \\ A_3 &= 2u_0 u_3 v_0 + 2u_1 u_2 v_0 + 2u_0 u_2 v_1 + u_1^2 v_1 \\ &\quad + 2u_0 u_1 v_2 + u_0^2 v_3 + 3v_0^2 v_3 + 6v_0 v_1 v_2 + v_1^3, \\ A_4 &= v_0 u_2^2 + 2u_0 v_0 u_4 + 2v_0 u_1 u_3 + 2u_0 v_1 u_3 \\ &\quad + 2u_1 v_1 u_2 + 2u_0 v_2 u_2 + u_1^2 v_2 + 2u_0 v_1 u_3 + u_0^2 v_4 \\ &\quad + 3v_0^2 v_4 + 6v_0 v_1 v_3 + 3v_0 v_2^2 + 3v_1^2 v_2, \end{aligned} \quad (26)$$

and

$$\begin{aligned} B_0 &= v_0^2 u_0 + u_0^3, \\ B_1 &= 2v_0 v_1 u_0 + v_0^2 u_1 + 3u_0^2 u_1, \\ B_2 &= 2v_0 v_2 u_0 + v_1^2 u_0 + 2v_0 v_1 u_1 + v_0^2 u_2 \\ &\quad + 3u_0^2 u_2 + 3u_0 u_1^2, \\ B_3 &= 2v_0 v_3 u_0 + 2v_1 v_2 u_0 + 2v_0 v_2 u_1 + v_1^2 u_1 + 2v_0 v_1 u_2 \\ &\quad + v_0^2 u_3 + 3u_0^2 u_3 + 6u_0 u_1 u_2 + u_1^3, \\ B_4 &= u_0 v_2^2 + 2v_0 u_0 v_4 + 2u_0 v_1 v_3 + 2v_0 u_1 v_3 + 2v_1 u_1 v_2 \\ &\quad + 2v_0 u_2 v_2 + v_1^2 u_2 + 2v_0 u_1 v_3 + v_0^2 u_4 + 3u_0^2 u_4 \\ &\quad + 6u_0 u_1 u_3 + 3u_0 u_2^2 + 3u_1^2 u_2. \end{aligned} \quad (27)$$

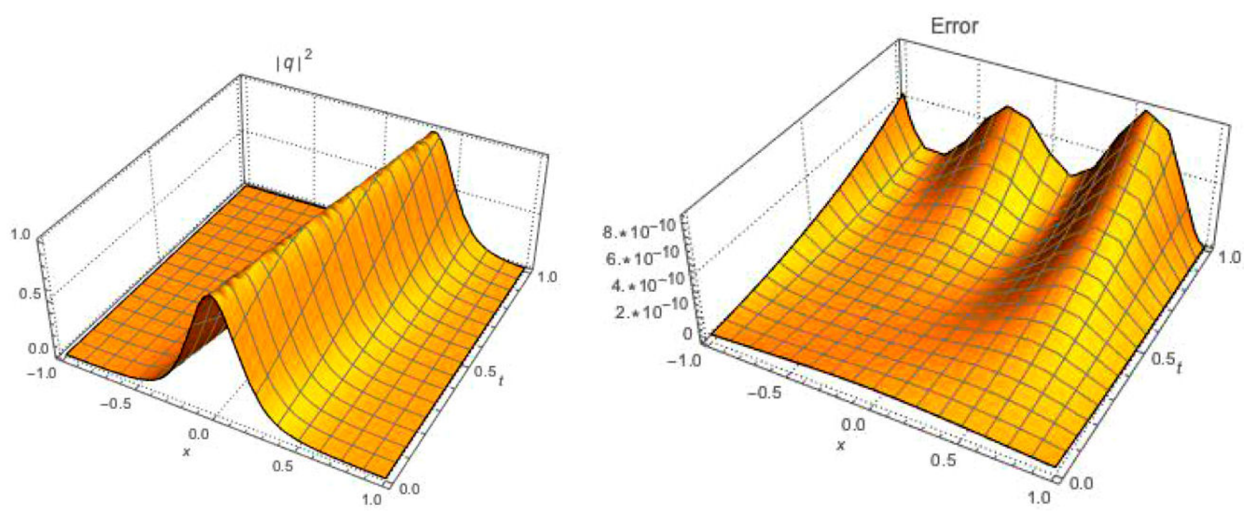

FIGURE 1. Case 1: Numerically computed profile (left) and absolute error (right).

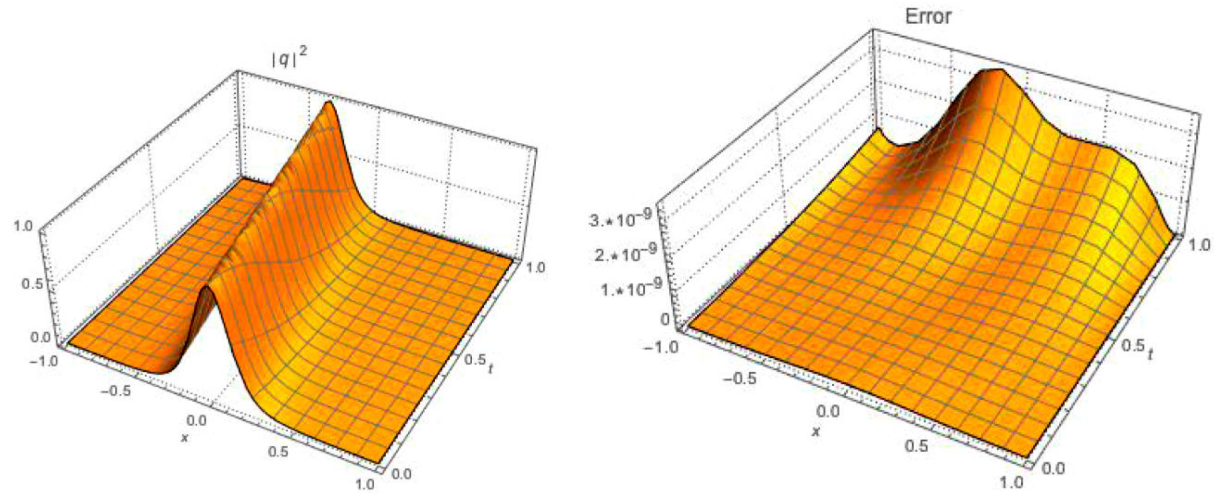

FIGURE 2. Case 2: Numerically computed profile (left) and absolute error (right).

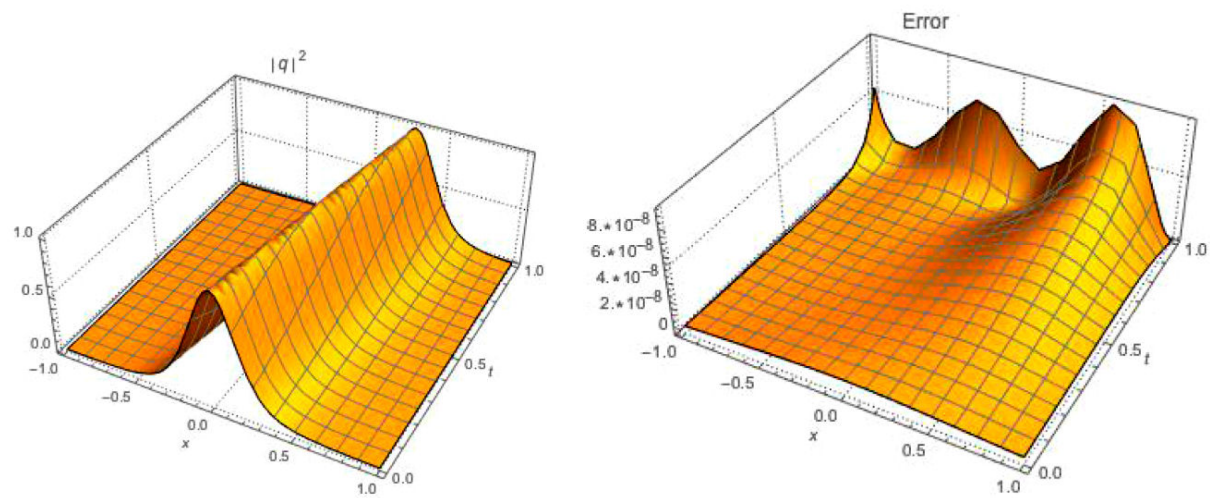

FIGURE 3. Case 3: Numerically computed profile (left) and absolute error (right).

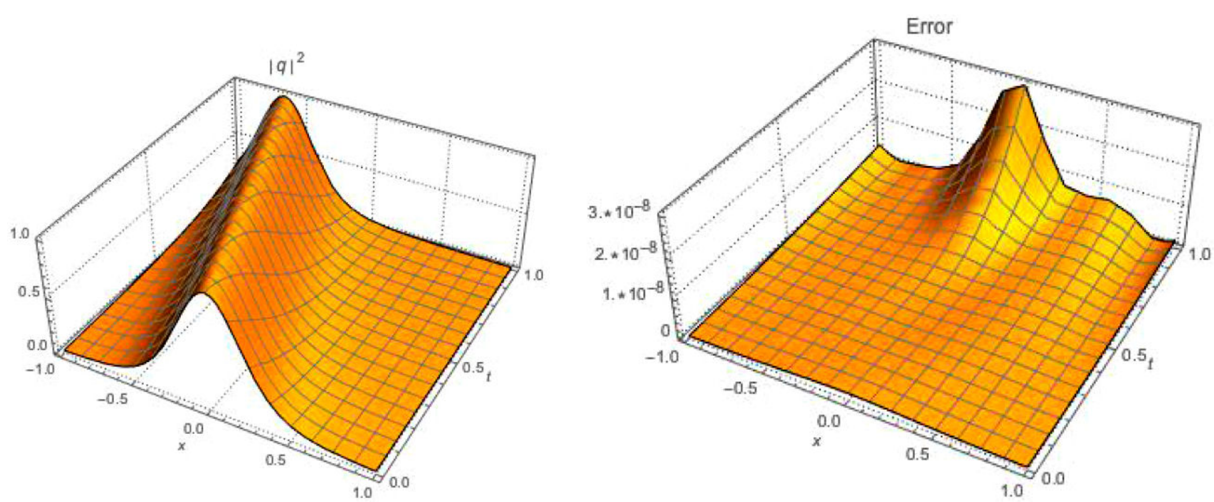

FIGURE 4. Case 4: Numerically computed profile (left) and absolute error (right).

TABLE I. Coefficients of Eq. (1) for bright solitons.

Cases	a_1	a_2	a_3	a_4	a_5	a_6	b	κ	ω	θ_0	ν	N	|Max Error|
1	1.2	0.01	0.024	0.50	4.20	0.33	-1.20	0.01	0.3	1.2	1.19	15	8.0×10^{-10}
2	1.5	0.07	0.058	0.25	3.20	0.16	-1.10	0.03	0.1	1.4	1.49	15	3.0×10^{-9}
3	1.0	0.09	0.013	0.12	1.00	0.08	-1.00	-0.02	0.7	1.6	1.00	12	8.0×10^{-8}
4	1.6	0.04	-0.030	0.80	1.20	0.10	-1.30	-0.01	0.3	1.3	1.60	12	3.0×10^{-8}

Finally, in conjunction with Eq. (24) and Eq. (25), all components of $u(x,t)$ in Eq. (17) will be easily determined; therefore, the complete solution $u(x,t)$ in Eq. (17) can be formally established. LADM provides a reliable technique that requires less work if compared with traditional techniques.

4. Numerical simulations

To illustrate the ability, reliability, and accuracy of the proposed method to find solutions of Eq. (1) in the case of bright solitons, some examples are provided. The results reveal that the method is very effective and simple. We now consider the initial condition at $t = 0$ from Eq. (2)

$$q(x,0) = A\Big(\operatorname{sech}[B(x)] + \operatorname{sech}^3[B(x)]\Big)e^{i[-\kappa x+\theta_0]}. \quad (28)$$

We now perform the simulation of the four cases listed in Table I, and the results and the respective absolute errors are shown in Figs. 1, 2, 3, and 4.

5. Conclusions

This paper addressed highly dispersive optical solitons, with Kerr law nonlinearity, by the aid of the Laplace-Adomian decomposition scheme. The focus was on bright solitons. The numerical results supplemented the analytical results, there were reported earlier, and the agreement is to a T. The error analysis was also profoundly impressive, as well. This shows extreme promise of the numerical algorithm that has been implemented in this paper. Thus, the results of this paper will be extended to additional laws of the nonlinear refractive index. These are cubic-quartic law, polynomial law, nonlocal nonlinearity, and others. Additionally, the results will be extended to birefringent fibers. The results of such research activities are on the horizon and are soon going to be made visible.

Acknowledgments

This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia under grant no. (KEP-15-130-40). The authors, therefore, acknowledge with thanks to DSR technical and financial support.

1. N. A. Kudryashov, Construction of nonlinear differential equations for description of propagation pulses in optical fiber, *Optik* **192** (2019) 162964, `https://doi.org/10.1016/j.ijleo.2019.162964`.
2. N. A. Kudryashov, Solitary and periodic waves of the hierarchi for propagation pulse in optical fiber, *Optik* **194** (2019) 163060, `https://doi.org/10.1016/j.ijleo.2019.163060`.
3. A. Biswas, A. H. Kara, A. S. Alshomrani, M. Ekici, Q. Zhou, and M. R. Belic, Conservation laws for highly dispersive optical solitons, *Optik* **199** (2019) 163283, `https://doi.org/10.1016/j.ijleo.2019.163283`.
4. H. U. Rehman, N. Ullah, and M. A. Imran, Highly dispersive optical solitons using Kudryashov's method, *Optik* **199** (2019) 163349, `https://doi.org/10.1016/j.ijleo.2019.163349`.
5. O. Pavón-Torres, M. A. Agüero, T. L. Belyaeva, A. Ramirez, and V. N. Serkin, Unusual self-spreading or self-compression of the cubic-quintic NLSE solitons owing to amplification or absorption, *Optik* **184** (2019) 446, `https://doi.org/10.1016/j.ijleo.2019.04.103`.
6. A. Biswas, M. Ekici, A. Sonmezoglu, and M. R. Belic, Highly dispersive optical solitons with Kerr law nonlinearity by F-expansion, *Optik* **181** (2019) 1028, `https://doi.org/10.1016/j.ijleo.2018.12.164`.
7. R. W. Kohl, A. Biswas, M. Ekici, Q. Zhou, S. Khan, A. S. Alshomrani, and M. R. Belic, Highly dispersive optical soliton perturbation with Kerr law by semi-inverse variational principle, *Optik* **199** (2019) 163226, `https://doi.org/10.1016/j.ijleo.2019.163226`.
8. A. Biswas, M. Ekici, A. Sonmezoglu, M. R. Belic, Highly dispersive optical solitons with Kerr law nonlinearity by extended Jacobi's elliptic function expansion, *Optik* **183** (2019) 395, `https://doi.org/10.1016/j.ijleo.2019.02.050`.

9. G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Method* (Springer, Dordrecht, 1994), https://doi.org/10.1007/978-94-015-8289-6.

10. S. A. Khuri, A Laplace decomposition algorithm applied to a class of nonlinear differential equations, *J. Appl. Math*. **1** (2001) 141, https://doi.org/10.1155/S1110757X01000183.

11. R. Rach, A convenient computational form for the Adomian polynomials, *J. Math. Anal. Appl.* **102** (1984) 415, https://doi.org/10.1016/0022-247X(84)90181-1.

12. A.-M. Wazwaz, A new algorithm for calculating adomian polynomials for nonlinear operators, *Appl. Math. Comput*. **111** (2000) 33, https://doi.org/10.1016/S0096-3003(99)00063-6.