

PAPER • OPEN ACCESS

Forces on membranes with in-plane order

To cite this article: G Barrientos *et al* 2017 *J. Phys. Commun.* **1** 045017

View the [article online](#) for updates and enhancements.

Related content

- [Dynamics of a particle on an axisymmetric surface under gravitational force: a geometric approach](#)
G Torres-Vargas
- [Stresses in lipid membranes](#)
R Capovilla and J Guven
- [Second variation of the Helfrich–Canham Hamiltonian and reparametrization invariance](#)
R Capovilla and J Guven



PAPER

Forces on membranes with in-plane order

OPEN ACCESS

RECEIVED
5 May 2017

REVISED
21 September 2017

ACCEPTED FOR PUBLICATION
4 October 2017

PUBLISHED
21 November 2017

Original content from this work may be used under the terms of the [Creative Commons Attribution 3.0 licence](#).

Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.



G Barrientos¹, G Chacón-Acosta², O González-Gaxiola²  and J A Santiago^{2,3} 

¹ Posgrado en Ciencias Naturales e Ingeniería, Universidad Autónoma Metropolitana-Cuajimalpa, Av. Vasco de Quiroga 4871, Ciudad de México 05348, Mexico

² Departamento de Matemáticas Aplicadas y Sistemas, Universidad Autónoma Metropolitana Cuajimalpa, Vasco de Quiroga 4871, Ciudad de México 05348, Mexico

³ Author to whom any correspondence should be addressed

E-mail: gbarrientos.pcni@dcniuamc.com, gchacon@correo.cua.uam.mx, ogonzalez@correo.cua.uam.mx and jsantiago@correo.cua.uam.mx

Keywords: fields on curved surfaces, nematic membranes, orientational order on curved membranes

Abstract

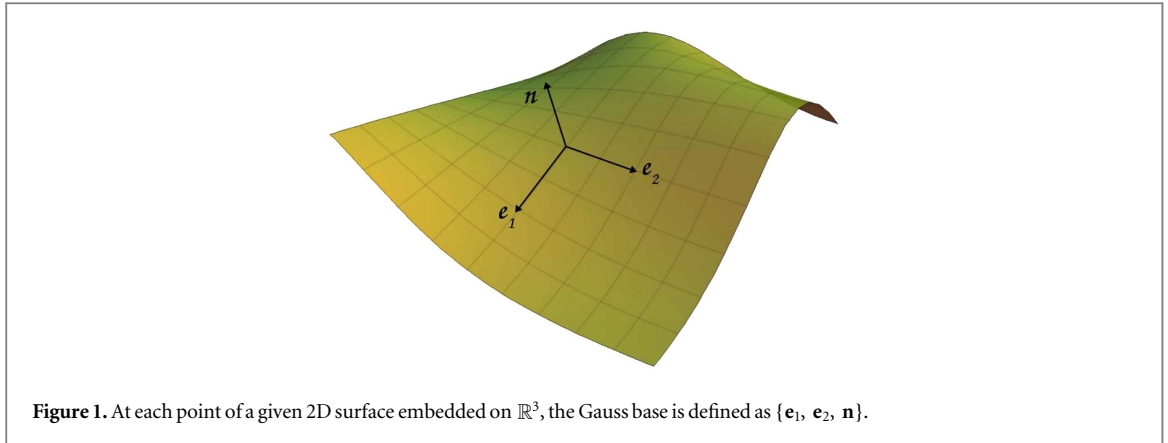
An ordered configuration of a vector field on an elastic curved surface induces a long-range interaction with the Gaussian curvature of the surface through the Green function of the Laplace–Beltrami operator of the surface. This in-plane orientational order configuration gives rise to a stress inducing a competition between this effective interaction and the surface tension. In this work we obtain the shape equation of the model from the surface tension term, that is a simpler way than those previously presented in the literature. We show that the stress and therefore the induced force, is a vector field tangent to the surface whose magnitude is given by the stiffness constant times the gaussian curvature of the membrane. We found that the shape of the membrane can be either a minimal surface, a developed surface or a hyperbolic one, depending on the particular regime. Explicit results are shown in the case of the catenoid and the helicoid. Finally, we find the self-adjoint stability operator of the model and obtain its eigenvalues for the nematic catenoid.

1. Introduction

Oriented configurations of vector fields on surfaces and its consequences have been of great interest over the years [1, 2]. This is due to vector fields over surfaces are models of two-dimensional systems such as liquid crystals, superfluids, and fluid membranes [3–6]. Although the description of thermal fluctuations is interesting, there is also interest in elucidating the limit of these phenomena in the case of zero temperature. For instance, some classical configurations of fields are only possible if the surface is curved, as for the well known phenomenon of geometric frustration [7]. In order to reduce the energy cost of this geometric frustration, the field induces topological defects in its configuration. The presence of these topological defects has the effect of shielding the Gaussian curvature of the surface. There is a force on the surface that appears as a consequence of the curvature of the membrane and causes that the surface acquires between other morphologies, the shape of cylindrical micelles, of tetrahedral and ellipsoidal vesicles [8].

Fluctuations in membranes with hexatic order were considered in the classical work by Nelson and Peliti [9]. In [10], research about equilibrium of nematic membranes was considered. In [11] authors deal with alternative forms of free energies for nematic membranes. Hexatic undulations modes in curved geometries have been studied in [12]. Patterns on curved surfaces of lamellar structures have been studied in [13]. In [14] authors apply conformal mapping to analyze forces on curved surfaces. In [16] several very interesting topological results about non-orientable nematics membranes were presented. Recently, in [15] the authors provide explicit formulas for the curvature associated with splay and bend fields in two-dimensions and relate them with a geometric frustration problem.

In these models the energy associated with ordered fields can be split into two parts. The first part is a bulk term that involves Gaussian curvature mediated by the Green function of the Laplace–Beltrami operator on the surface. The second term involves topological defects and also depends on the global shape of the surface



through the geometric potential evaluated at the point defects [7]. Far from the points where topological defects are located, the in-plane orientational order induce a force that goes against the elastic forces induced by the surface itself, either surface tension or elastic bending forces.

In this work, we obtain the shape equation of the membrane $K(\sigma + \kappa\mathcal{R}) = 0$, which was previously obtained in [17]. σ is the surface tension, κ the stiffness constant, K and \mathcal{R} the mean and scalar curvature of the surface respectively. Only normal deformations of the surface play a role, this is a consequence of the energy invariance under surface reparameterizations. Thus, the effect of the tangential deformations will be evident in the boundary terms Minimal surfaces $K = 0$ remain as solutions of the shape equations if the Canham–Helfrich bending energy is added but if neglected, the corresponding solutions are hyperbolic-like surfaces $\mathcal{R} = -\sigma/\kappa$, [17, 18]. In the hyperbolic case, the force coming from the in-plane order cancels out at every point on the membrane with the force coming from the surface tension. In the in-plane ordering regime we also found that developed surfaces $\mathcal{R} = 0$, are solutions of the shape equations as well.

From the boundary terms, through translational invariance of the energy we can identify the conserved Noether charge [20]. We obtain that the associated stress can be written as $\mathbf{f}^a = -g^{ab}(\sigma + \kappa\mathcal{R})\mathbf{e}_b$. A term proportional to the scalar curvature appears. We note that there is no component of the stress in the outward direction to the membrane, which is what one expects from purely intrinsic energy. Hence, in the case of hyperbolic-like membranes the effective surface tension vanishes. On minimal surfaces the scalar curvature is not a constant and the behavior of the stress does depend on the point where we evaluate. In a generic model, rotational symmetry generates an orbital and also an intrinsic torque. In such a case, as there is no component in the normal direction of the stress tensor, we do not have intrinsic torque either. In order to illustrate the results both the force and torque on the catenoid and the helicoid are studied.

The paper is organized as follows, in section 2 we give a brief introduction to the geometric description of curved surfaces. We also give a short presentation of the in-plane ordering energy starting with the Frank energy for liquid crystals on a curved surface. Although this information has been included in several previous works, this will be useful to establish definitions and notation along the paper. In section 3 we obtain the response of the energy to infinitesimal deformations of the embedding function. Then we identify the Euler–Lagrange derivative as well as the boundary terms The in-plane energy is written in isothermal coordinates so that the information of the curvature and its deformations is encoded in the conformal factor. The stress tensor and the torque are obtained in section 4. As a classical example of a minimal surface, in section 5 we present some results on a catenoid and the helicoid is briefly considered. The in-plane force takes its maximum value at the waist of the catenoid and decrease as we move away. Finally, we obtain the stability operator of the energy in section 6. In section 7 summary and some conclusions are presented.

2. Geometric preliminaries

Let us consider a two-dimensional surface embedded in the three-dimensional euclidean space \mathbb{R}^3 , with $\mathbf{x} = (x^1, x^2, x^3)$. We can write an explicit equation for the embedding as $\mathbf{x} = \mathbf{X}(\xi^a)$ parametrized by coordinates ξ^a , ($a = 1, 2$). The induced metric on the surface is given by $g_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b$, where dot stands for the inner product in \mathbb{R}^3 and $\mathbf{e}_a = \partial_a \mathbf{X}$ are the tangent vectors to the surface. The unit vector field normal to the surface is defined as $\mathbf{n} = (\mathbf{e}_1 \times \mathbf{e}_2) / \sqrt{g}$. Therefore, these three vectors form a right-hand base as can be seen in figure 1. Partial derivatives of the tangent vectors of the surface are described by the Gauss equation

$$\partial_a \mathbf{e}_b = -K_{ab} \mathbf{n} + \Gamma_{ab}^c \mathbf{e}_c, \quad (1)$$

where K_{ab} is a second order symmetric tensor called the extrinsic curvature of the surface (also known as the second fundamental form), Γ_{bc}^a are the Christoffel symbols needed to define the covariant derivative ∇_a compatible with the induced metric such that $\nabla_a g_{bc} = 0$. Using this derivative the Gauss equation can be rewritten simply as $\nabla_a \mathbf{e}_b = -K_{ab} \mathbf{n}$. As a consequence of the integrability conditions the extrinsic and intrinsic curvature of the surface are related by the well-known Gauss–Codazzi–Mainardi equations [21]

$$KK_{ab} - K_a^c K_{cb} = R_{ab}, \quad (2)$$

$$\nabla_b K^b_a = \nabla_a K. \quad (3)$$

The inverse of the induced metric g^{ab} is used in these equations to rise tangential indices, then $K^a_b = g^{ac} K_{cb}$ and so on. Also $K = g^{ab} K_{ab}$ is the mean curvature of the surface and $\mathcal{R} = g^{ab} \mathcal{R}_{ab}$ is the scalar curvature. Thus, we can then obtain the scalar version of equation (2) as

$$K^2 - K^{ab} K_{ab} = \mathcal{R}. \quad (4)$$

The Ricci tensor of the surface $\mathcal{R}_{ab} = \mathcal{R}^d_{adb}$ is a contraction of the Riemann tensor defined as

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) A^c = \mathcal{R}^c_{dab} A^d. \quad (5)$$

In the case of two-dimensional surfaces we know that $\mathcal{R}_{ab} = g_{ab} \mathcal{R}/2$.

With the covariant derivative we can write the Laplace–Beltrami operator of the surface as follows

$$\nabla_g^2 f = g^{ab} \nabla_a \nabla_b f. \quad (6)$$

Instead of using generalized coordinates ξ^a we can parameterize the surface with the so-called isothermal coordinates u^a . The advantage of this coordinates is that the metric is conformally flat

$$g_{ab} = w \delta_{ab}, \quad (7)$$

where w is the conformal factor. In such coordinates the mean curvature can be conveniently written as

$$K = -\frac{1}{w} \mathbf{n} \cdot \nabla^2 \mathbf{X} = -\mathbf{n} \cdot \nabla_g^2 \mathbf{X}, \quad (8)$$

where ∇^2 is the euclidean laplacian in the isothermal u^a coordinates, and the Laplace–Beltrami operator ∇_g^2 is just $w^{-1} \nabla^2$ in these coordinates. It is not difficult to write the curvature scalar as

$$\mathcal{R} = -\frac{\nabla^2 \log w}{w}. \quad (9)$$

Let us consider a unit vector field defined on the tangent plane to the surface, $\mathbf{m} = m^a \mathbf{e}_a$, then $\mathbf{m} \cdot \mathbf{m} = g_{ab} m^a m^b = 1$. The splay and bending energy of the field are modeled by the Frank energy which on two-dimensional curved surfaces is given by

$$F = \frac{1}{2} \int dA [K_1 (\nabla_a m^a)^2 + K_3 (m^a \nabla_a m_b)^2]. \quad (10)$$

Here the field m^a describes the ordering direction of the nematic phase. When the coefficients satisfy $K_1 = K_3 = \kappa$, equation (10) simplifies to

$$F = \frac{\kappa}{2} \int dA (\nabla_a m^b)^2, \quad (11)$$

where $dA = \sqrt{g} d^2 \xi$ is the area element.

On the other hand, the energy density \mathcal{F} can be expanded in terms of the extrinsic curvature as follows [6]

$$\begin{aligned} \mathcal{F} &= g^{ab} (\partial_a \mathbf{m}) \cdot (\partial_b \mathbf{m}), \\ &= g^{ab} \nabla_a m^c \nabla_b m_c + m^c K_{ac} K_d^a m^d, \end{aligned} \quad (12)$$

where the Gauss equation was used. The first term in equation (12) is a contribution intrinsic to the surface, while the second one has a quadratic coupling with the extrinsic curvature K_{ab} of the surface. By using equation (2) we can write

$$\begin{aligned} \mathcal{F} &= (\nabla_a m^b)^2 + m^a K K_{ab} m^b - m^a \mathcal{R}_{ab} m^b, \\ &= (\nabla_a m^b)^2 + m^a K K_{ab} m^b - \frac{\mathcal{R}}{2}. \end{aligned} \quad (13)$$

In the second line of equation (13) we used the fact that the Einstein tensor $\mathcal{G}_{ab} = \mathcal{R}_{ab} - (g_{ab}/2) \mathcal{R} = 0$, for two-dimensional surfaces. When integrated, the last term is a boundary term consequence of the Gauss–Bonnet theorem. When the surface is a sphere, we note that only the intrinsic energy becomes relevant since the extrinsic energy is a contribution to the surface tension. Some extrinsic effects in the Frank energy have also been studied [19, 22, 28].

If the direction of m^a is along the normal vector to the layers then the second term in equation (10) takes its minimal energy along geodesics, i.e. $m^a \nabla_a m^b = 0$. That is, the layers are normal to the geodesic of the surface. The extrinsic energy density can be identified as $KK_{ab} m^a m^b = K\kappa_n$, where κ_n is the normal curvature of the geodesics [21]. It has a minimum along asymptotic curves on the surface where $\kappa_n = 0$. On a cylindrical surface, for instance, this energy vanishes if the director is aligned with the cylinder's axis.

This field theory can be alternatively described in terms of the spin connection $\Omega = e^a \Omega_a$ which is a vector function defined in the tangent space of the surface [23], and is related with gaussian curvature as follows

$$\nabla \times \Omega = \frac{\mathcal{R}}{2} \mathbf{n}. \quad (14)$$

By using the orthonormal frame ϵ_α with $\alpha = \{1, 2\}$, the field \mathbf{m} is written in terms of the angle Θ with ϵ_1 as

$$\begin{aligned} \mathbf{m} &= m^\alpha \epsilon_\alpha, \\ &= \cos \Theta \epsilon_1 + \sin \Theta \epsilon_2. \end{aligned} \quad (15)$$

The intrinsic energy can be written as a function of this angle

$$F = \frac{\kappa}{2} \int dA g^{ab} (\partial_a \Theta - \Omega_a) (\partial_b \Theta - \Omega_b). \quad (16)$$

For a cylindrical surface this energy reaches a minimal value for $\Theta = \text{Const.}$, as horizontal loops.

As it is well known along in-plane ordering configuration, this energy can be written as a non-local interaction of the Gaussian curvature mediated by the Green function of the laplacian defined on the surface [6]. By adding this to the bending energy of the membrane and taking into account the topological charges q_i , the energy is given by

$$H = \sigma \int dA + \frac{\bar{\kappa}}{2} \int dA K^2 + \frac{\kappa}{2} \int \int dA dA' \rho(\xi) G(\xi, \xi') \rho(\xi'), \quad (17)$$

where $\bar{\kappa}$ is the bending rigidity, $\rho(\xi) = \mathcal{R}(\xi) - S(\xi)$ involves the scalar curvature at the point ξ , $S(\xi) = \frac{1}{\sqrt{g}} \sum_i q_i \delta(\xi, \xi_i)$ stands for the charges and $G(\xi, \xi')$ is the Green function of the operator $-\nabla_g^2 = -g^{ab} \nabla_a \nabla_b$ on the surface. Without the topological charges this energy is known as the Liouville field theory in quantum gravity [24]. In the next section we obtain the infinitesimal deformations of this energy (neglecting the bending energy) and identify the shape equations.

3. Shape equation and boundary conditions

To find the shape equations we track the response of the energy in equation (17) (neglecting the bending energy) to small deformations of the embedding functions, $\mathbf{X} \rightarrow \mathbf{X} + \delta \mathbf{X}$. We project the deformation in the Gauss frame as [27]

$$\begin{aligned} \delta \mathbf{X} &= \delta_{\parallel} \mathbf{X} + \delta_{\perp} \mathbf{X}, \\ &= \Phi^a \mathbf{e}_a + \Phi \mathbf{n}. \end{aligned} \quad (18)$$

Under this deformation one can calculate the corresponding deformation of each geometric function defined on the surface. Particularly relevant are the perpendicular deformations that have been calculated in detail [27]. The deformation of the induced metric is

$$\delta g_{ab} = 2K_{ab} \Phi + \nabla_a \Phi_b + \nabla_b \Phi_a. \quad (19)$$

With this result the area element of the surface deforms as follows $\delta \sqrt{g} = \sqrt{g} (\nabla_a \Phi^a + K \Phi)$. In isothermal coordinates we have that the deformation of the conformal factor is given by $\delta \log w = (\nabla_a \Phi^a + K \Phi)$. The tangential deformation of the conformal factor is identified as

$$\delta_{\parallel} \log w = \nabla_a \Phi^a = \frac{1}{w} \partial_a (w \Phi^a), \quad (20)$$

whereas the normal deformation of the conformal factor can be written as

$$\delta_{\perp} \log w = K \Phi. \quad (21)$$

We can now find the deformations of the scalar curvature. From its definition equation (9), we have

$$\begin{aligned} \delta_{\perp} \mathcal{R} &= -K \mathcal{R} \Phi - \frac{\nabla^2 (K \Phi)}{w}, \\ \delta_{\parallel} \mathcal{R} &= -\mathcal{R} \nabla_a \Phi^a - \frac{\nabla^2 (\nabla_c \Phi^c)}{w}, \end{aligned} \quad (22)$$

and thus we can get

$$\begin{aligned}\delta_{\perp}(\sqrt{g}\mathcal{R}) &= -\nabla^2(K\Phi), \\ \delta_{\parallel}(\sqrt{g}\mathcal{R}) &= -\nabla^2(\nabla_c\Phi^c).\end{aligned}\quad (23)$$

With these elements we can find the shape equations of the model. Write the deformation of the in-plane order energy

$$\delta F = \kappa \delta \int dA \rho(\xi) \mathcal{U}(\xi), \quad (24)$$

where the function $\mathcal{U}(\xi)$ defined by

$$\mathcal{U}(\xi) = \int dA' G(\xi, \xi') \rho(\xi'), \quad (25)$$

has been introduced. A technical problem for this calculation is to find the deformation of Green function. One way is to use the switch with the Laplacian to pass the deformation to the Green function. This will be addressed in an upcoming note. Here we use an idea based in a recent work by Giomi [17]. Let us write the energy in isothermal coordinates such that the area element turns to $dA = w d^2u$, then for δF we have

$$\delta F = \kappa \int d^2u \delta(w \rho) \mathcal{U}(u). \quad (26)$$

In these coordinates we can write

$$w\rho = -\nabla^2 \log w - \sum_i q_i \delta(u, u_i), \quad (27)$$

such that $\delta(w\rho)$ involves only deformations in equation (23). We then obtain the normal deformation of the energy that includes the surface tension as

$$\delta_{\perp}F = \int dA \mathcal{E}\Phi + \kappa \int d^2u \partial_a[\Phi K \partial_a \mathcal{U} - \mathcal{U} \partial_a(K\Phi)], \quad (28)$$

where the Euler–Lagrange derivative is given by

$$\mathcal{E} = (\sigma + \kappa\rho)K, \quad (29)$$

far from the topological defect positions $\rho = \mathcal{R}$, as was anticipated in [17]. From equation (29), we see that minimal surfaces $K = 0$ and hyperbolic-like surfaces $\mathcal{R} = -\sigma/\kappa$, are solutions of the shape equations $K(\sigma + \kappa\mathcal{R}) = 0$, [17]. Different types of solutions appear by considering the regime where the surface tension is negligible. In such a case developable surfaces also appear as solutions. As a consequence of invariance under translations the tangential deformation of the energy is a boundary term. To see this explicitly we write

$$\delta_{\parallel}F = \int dA (\sigma + \kappa\rho) \nabla_a \Phi^a + \kappa \int d^2u \partial_a[\nabla_c \Phi^c \partial_a \mathcal{U} - \mathcal{U} \partial_a(\nabla_c \Phi^c)]. \quad (30)$$

In the first term we can integrate by parts to obtain

$$\int dA (\sigma + \kappa\rho) \nabla_a \Phi^a = \int dA \nabla_a[\Phi^a(\sigma + \kappa\rho)] - \kappa \int dA \Phi^a \nabla_a \rho. \quad (31)$$

Due to the relation

$$\sqrt{g} \Phi^a \nabla_a \rho = \Phi^a \nabla_a(\sqrt{g}\rho) = \delta_{\parallel}(\sqrt{g}\rho), \quad (32)$$

we can write the second integral in equation (31) as

$$\int dA \Phi^a \nabla_a \rho = \delta_{\parallel} \int dA \rho = \delta_{\parallel} \int dA \left(\mathcal{R} - \frac{1}{\sqrt{g}} \sum_i q_i \delta(\xi, \xi_i) \right) = 0. \quad (33)$$

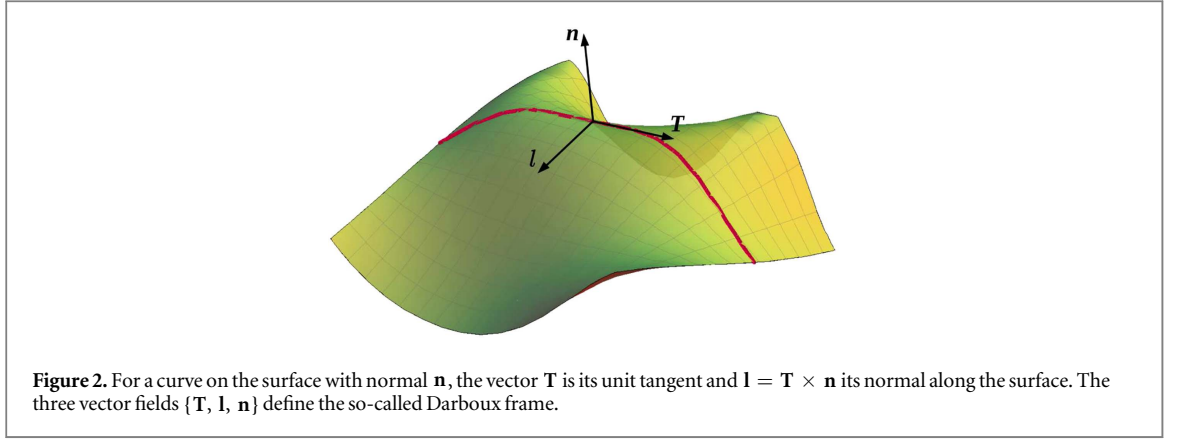
This integral vanishes because the first term is the Gauss–Bonnet invariant and the second one is the total defect's charge, that is a constant. Therefore, the total deformation of the energy $\delta F = \delta_{\perp}F + \delta_{\parallel}F$ is given by

$$\delta F = \int dA \mathcal{E}\Phi + \int dA \nabla_a S_1^a + \int dA \nabla_a S_2^a, \quad (34)$$

where we have defined the boundary terms

$$\begin{aligned}S_1^a &= \Phi^a(\sigma + \kappa\rho), \\ S_2^a &= (\nabla_c \Phi^c + K\Phi) \partial^a \mathcal{U} - \mathcal{U} \partial^a(\nabla_c \Phi^c + K\Phi).\end{aligned}\quad (35)$$

For the case of a membrane without a boundary these terms vanish. Note that S_2^a involves the deformation of the membrane's area, that is



$$\delta A = \int dA (K\Phi + \nabla_a \Phi^a). \quad (36)$$

This means that if the mass density of the membrane remains constant under deformations, then $K\Phi + \nabla_a \Phi^a = 0$, even at the boundary. In the presence of a boundary we can add the term

$$F_b = \sigma_b \oint_C ds, \quad (37)$$

where σ_b is the linear tension of the membrane's boundary. Deformation of the boundary term is found to be

$$\delta \oint_C ds = \oint_C ds (\kappa_g l_a \Phi^a + \kappa_n \Phi), \quad (38)$$

where κ_g is the geodesic curvature of the boundary and $\kappa_n = K_{ab} t^a t^b$ its normal curvature. At this point we recall the Darboux frame adapted to the boundary, $\mathbf{T} = t^a \mathbf{e}_a$ is the tangent vector field and $\mathbf{l} = l^a \mathbf{e}_a$ a normal vector to the curve such that $\mathbf{l} = \mathbf{T} \times \mathbf{n}$ (see figure 2). Accordingly, in this case we can write

$$\delta F = \int dA \mathcal{E} \Phi + \oint_C ds [(\sigma + \kappa \mathcal{R} + \sigma_b \kappa_g) l_a \Phi^a + \sigma_b \kappa_n \Phi]. \quad (39)$$

On hyperbolic-like surfaces with $\mathcal{R} = -\sigma/\kappa$, the boundary conditions are given by $\kappa_n = 0$ and $\kappa_g = 0$, [17, 18]. The boundary of the membrane is an asymptotic geodesic curve. If the bulk is a minimal surface the boundary terms involve the gaussian curvature. Then, in this case we have

$$\begin{aligned} \mathcal{R} &= -K^{ab} K_{ab}, \\ &= -(\kappa_t^2 + 2\tau_g^2 + \kappa_n^2), \end{aligned} \quad (40)$$

where we have defined $\kappa_t = K_{ab} l^a l^b$ and $\tau_g = K_{ab} t^a l^b$. Consequently, if the boundary curve is not fixed it satisfy the following conditions

$$\begin{aligned} \sigma - \kappa(\kappa_t^2 + 2\tau_g^2 + \kappa_n^2) + \sigma_b \kappa_g &= 0, \\ \sigma_b \kappa_n &= 0. \end{aligned} \quad (41)$$

4. Stress

Let us take an infinitesimal translation on the energy in equation (34). Then $\delta \mathbf{X} = \mathbf{a}$ which implies that $\Phi = \mathbf{a} \cdot \mathbf{n}$ and $\Phi^a = g^{ab} \mathbf{a} \cdot \mathbf{e}_b$. In that case $S_2^a = 0$ and therefore the only contribution comes from the S_1^a term. We have

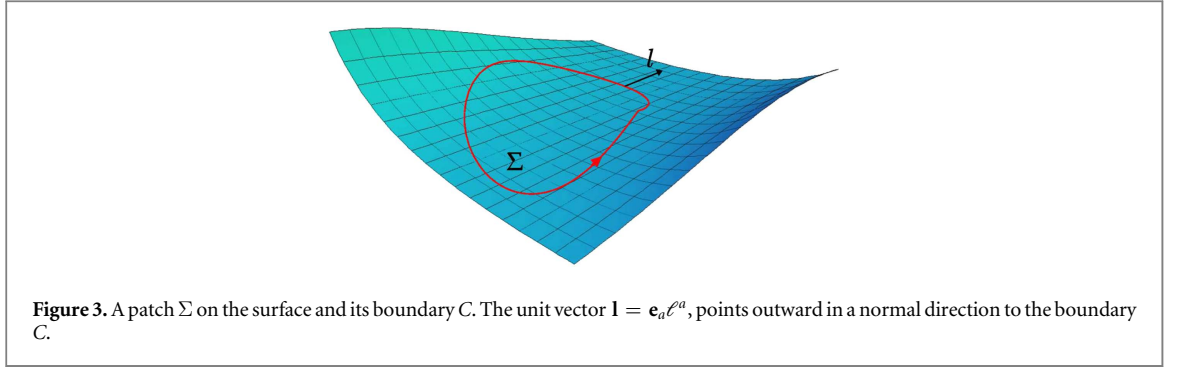
$$\delta F = \mathbf{a} \cdot \int dA (\mathcal{E} \mathbf{n} - \nabla_a \mathbf{f}^a), \quad (42)$$

where the stress is given by the vector field

$$\mathbf{f}^a = -g^{ab} (\sigma + \kappa \rho) \mathbf{e}_b, \quad (43)$$

which is tangent to the membrane. It includes (a constant) surface tension, and the term $\kappa \rho$ consequence of the in-plane order. Invariance under translations implies that

$$\nabla_a \mathbf{f}^a = \mathcal{E} \mathbf{n}. \quad (44)$$



On equilibrium shape solutions $\mathcal{E} = 0$ and the conservation law $\nabla_a \mathbf{f}^a = 0$ follows. From equation (42) we have

$$\delta F = -\mathbf{a} \cdot \int dA \nabla_a \mathbf{f}^a. \quad (45)$$

In this manner we can identify the force acting on the boundary C of the patch Σ , [20]:

$$\begin{aligned} \mathbf{F} &= \int_{\Sigma} dA \nabla_a \mathbf{f}^a = \oint_C ds \ell_a \mathbf{f}^a, \\ &= -\sigma \oint_C ds \ell \left(1 + \frac{\kappa}{\sigma} \rho \right), \end{aligned} \quad (46)$$

where $\ell = \mathbf{e}_a \ell^a$ is a vector field tangent to the membrane and in the normal direction to the loop C (see figure 3).

Additionally to the surface tension, a force due to the in-plane ordered phase that is proportional to the curvature of the membrane emerges. Depending on the curvature \mathcal{R} this force increases or decreases the area of the patch Σ . On hyperbolic-like surfaces with scalar curvature $\mathcal{R} = -\sigma/\kappa$ the force coming of the in-plane ordered phase is cancelled with the surface tension.

An alternative route to find the stress tensor is using a general expression obtained introducing auxiliary variables [25]. Starting from the functional energy $H = \int dA \mathcal{H}$ one can define the tensors

$$T^{ab} = -\frac{2}{\sqrt{g}} \frac{\delta(\sqrt{g}\mathcal{H})}{\delta g_{ab}} \quad \text{and} \quad \mathcal{H}^{ab} = \frac{\delta \mathcal{H}}{\delta K_{ab}}. \quad (47)$$

In the basis $\{\mathbf{e}_a, \mathbf{n}\}$, we can write the vector field as

$$\mathbf{f}^a = f^{ab} \mathbf{e}_b + f^a \mathbf{n}, \quad (48)$$

where

$$f^{ab} = T^{ab} - \mathcal{H}^{ac} K_c^b, \quad \text{and} \quad f^a = -\nabla_b \mathcal{H}^{ab}. \quad (49)$$

In the case of the in-plane energy with surface tension σ , we see that it does not depend on the extrinsic curvature, so we have that $\mathcal{H}^{ab} = 0$. We can write the deformation in isothermal coordinates

$$\delta F = \frac{\sigma}{2} \int d^2u w g^{ab} \delta g_{ab} - \frac{\kappa}{2} \int d^2u \nabla^2 (g^{ab} \delta g_{ab}) \mathcal{U}(u). \quad (50)$$

After integrations by parts, neglecting boundary terms and using the fact that $\nabla^2 \mathcal{U} = -w\rho = -\sqrt{g} \rho$, the second term of the last equation can be written as

$$\frac{\kappa}{2} \int dA \rho g^{ab} \delta g_{ab}. \quad (51)$$

In this way only the symmetric tensor T^{ab} given by

$$T^{ab} = -g^{ab} \sigma \left(1 + \frac{\kappa}{\sigma} \rho \right), \quad (52)$$

contributes to the stress tensor. In the surface tension regime, that is when $0 < \kappa \mathcal{R} / \sigma < 1$, the surface's shape tends to a minimal surface with $T^{ab} = -g^{ab} \sigma$, as expected [17].

If the shape of the membrane is such that its scalar curvature is $\mathcal{R} = -\sigma/\kappa$, the T^{ab} tensor vanishes at every point of the membrane (except at the defect points). Note that the contracted tensor

$$T^{ab} g_{ab} = -2(\sigma + \kappa \rho), \quad (53)$$

also vanishes for both hyperbolic-like membranes and developable surfaces if the surface tension is negligible.

Let us now take an infinitesimal rotation $\delta\mathbf{X} = \mathbf{b} \times \mathbf{X}$, then we have that $\Phi = \mathbf{b} \cdot \mathbf{X} \times \mathbf{n}$ and $\Phi_a = \mathbf{b} \cdot \mathbf{X} \times \mathbf{e}_a$. Thus, we see again that $\nabla_a \Phi^a + K\Phi = 0$ and $S_2^a = 0$. Once again the only contribution comes from $S_1^a = \mathbf{b} \cdot \mathbf{X} \times \mathbf{f}^a$. So that the energy deforms as

$$\delta F = \mathbf{b} \cdot \int dA (\mathcal{E} \mathbf{X} \times \mathbf{n} - \nabla_a \mathbf{M}^a), \quad (54)$$

where the vector field

$$\mathbf{M}^a = -\mathbf{X} \times \mathbf{e}^a (\sigma + \kappa \rho), \quad (55)$$

is the torque about the origin $\mathbf{M}^a = \mathbf{X} \times \mathbf{f}^a$. Due to energy invariance under rotations we get

$$\nabla_a \mathbf{M}^a = \mathcal{E} \mathbf{X} \times \mathbf{n}, \quad (56)$$

so that the conservation law $\nabla_a \mathbf{M}^a = 0$, follows along the shape solutions of the Euler–Lagrange equations $\mathcal{E} = 0$. Under an infinitesimal rotation the energy deformation is

$$\begin{aligned} \delta F &= -\mathbf{b} \cdot \int_{\Sigma} dA \nabla_a \mathbf{M}^a = -\mathbf{b} \cdot \oint_C ds l_a \mathbf{M}^a, \\ &= -\mathbf{b} \cdot \oint_C ds \mathbf{X} \times \mathbf{l} (\sigma + \kappa \rho). \end{aligned} \quad (57)$$

In the next section the force and torque on the *nematic catenoid*, a minimal surface with in-plane order, are obtained.

5. Minimal surfaces: the catenoid

An illustrative example here we present the surface of the catenoid. It is a minimal surface that can be parametrized as a surface of revolution as follows

$$\mathbf{X}(z, \phi) = (a \cosh(z/a) \cos \phi, a \cosh(z/a) \sin \phi, z). \quad (58)$$

where a is the minimum radius at the waist, $\phi \in [0, 2\pi]$ and $z \in (-\infty, \infty)$. In these coordinates the infinitesimal distance is given by a metric conformal to \mathbb{R}^2 :

$$ds^2 = \cosh^2(z/a) (dz^2 + a^2 d\phi^2). \quad (59)$$

In terms of the radial variable $\rho = a \cosh(z/a)$ we can write it as

$$ds^2 = \frac{\rho^2}{\rho^2 - a^2} d\rho^2 + \rho^2 d\phi^2, \quad (60)$$

which in turn can be simply written as

$$ds^2 = dl^2 + (a^2 + l^2) d\phi^2, \quad (61)$$

where now the variable $l^2 = \rho^2 - a^2$ such that $l = a \operatorname{arcsinh}(z/a)$, has been introduced. In this way the parametrization of the surface can be rewritten as

$$\mathbf{X}(l, \phi) = (\sqrt{l^2 + a^2} \cos \phi, \sqrt{l^2 + a^2} \sin \phi, a \operatorname{arcsinh}(l/a)). \quad (62)$$

The tangential vectors are given by

$$\begin{aligned} \mathbf{e}_l &= \left(\frac{l}{\sqrt{l^2 + a^2}} \cos \phi, \frac{l}{\sqrt{l^2 + a^2}} \sin \phi, \frac{a}{\sqrt{l^2 + a^2}} \right), \\ \mathbf{e}_\phi &= (-\sqrt{l^2 + a^2} \sin \phi, \sqrt{l^2 + a^2} \cos \phi, 0), \end{aligned} \quad (63)$$

and the unit normal outward to the surface is

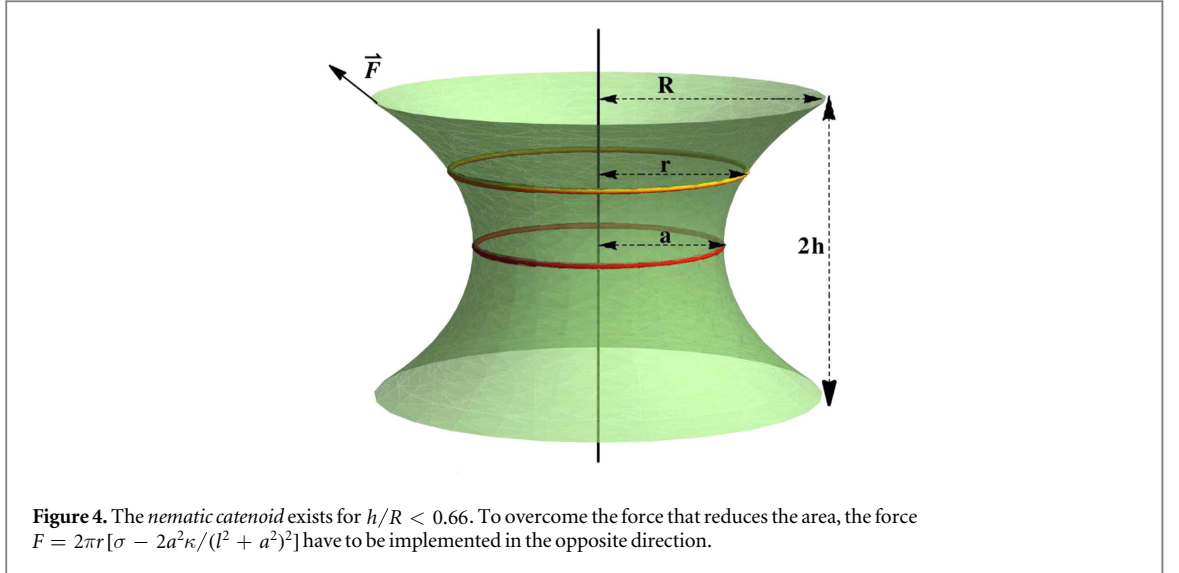
$$\mathbf{n} = \frac{1}{\sqrt{a^2 + l^2}} (a \cos \phi, a \sin \phi, -l). \quad (64)$$

The non-zero elements of the extrinsic curvature can be expressed in coordinates $\{l, \phi\}$ and are the following

$$K^{ll} = -\frac{a}{l^2 + a^2}, \quad K^{\phi\phi} = \frac{a}{(l^2 + a^2)^2}. \quad (65)$$

The scalar curvature is found to be

$$\mathcal{R} = -\frac{2a^2}{(a^2 + l^2)^2}. \quad (66)$$



The force acting on a line segment does not depend on the angular variable and is given by

$$\mathbf{F} = - \oint ds \mathbf{l} (\sigma + \kappa \mathcal{R}), \quad (67)$$

where

$$\sigma + \kappa \mathcal{R} = \sigma - \frac{2a^2\kappa}{(l^2 + a^2)^2}. \quad (68)$$

On the waist of the catenoid $l = 0$ and the force per unit length is just $\sigma - 2\kappa/a^2$. It decreases as we move away from the waist. Along the parallel such that $l = l_0$ the unit tangent is $\mathbf{T} = (-\sin \varphi, \cos \varphi, 0)$. The vector $\mathbf{l} = -1/\sqrt{l_0^2 + a^2} (l_0 \cos \varphi, l_0 \sin \varphi, a)$ complements the Darboux frame on this loop, [21]. When integrated, the radial component of the force is canceled and thus the total force on each horizontal loop is given by $\mathbf{F} = -(2\pi a)(\sigma - 2\kappa/a^2)\mathbf{k}$.

In the same way the torque acting on a segment of line is given by

$$\boldsymbol{\tau} = \mathbf{X} \times \mathbf{l} \left(\sigma - \frac{2a^2\kappa}{(l^2 + a^2)^2} \right), \quad (69)$$

on the waist of the catenoid it is given by

$$\boldsymbol{\tau} = - \left(\sigma - \frac{2\kappa}{a^2} \right) a (\sin \varphi, \cos \varphi, 0). \quad (70)$$

In this example the resulting force due to the ordering implies an effective surface tension that up to leading order is $\sigma_{\text{eff}} = \sigma - 2\kappa/a^2$. To obtain experimentally a catenoid of height $2h$ with a couple of rings with radius R each, we need a force that is tangential to the surface and at the same time perpendicular to the ring, whose magnitude is $F_c = 2\pi R \sigma_{\text{eff}}$. Where $h/R < 0.66$ (see figure 4). Note that on a loop of radius $r < R$ the force decreases as $F = 2\pi r \sigma_{\text{eff}} < F_c$. This gives an upper limit to the force as a function of h expressed as $F_c = 2\pi h \sigma_{\text{eff}} / 0.66$. If we pull harder catenoid degenerates into two films on the rings. In order to have a better description one should take into account the bending force. Induced stress due to bending is given by [25]

$$\mathbf{f}^a = \bar{\kappa}(K - K_0) \left[K^{ab} - \frac{1}{2}(K - K_0)g^{ab} \right] \mathbf{e}_b - \bar{\kappa}(\nabla^a K)\mathbf{n}, \quad (71)$$

that on a minimal surfaces with $K = 0$ becomes $\mathbf{f}^a = -\bar{\kappa}K_0(K^{ab} + \frac{1}{2}K_0g^{ab})\mathbf{e}_b$. On horizontal loops of the catenoid it is found that the bending induced force per unit length at a height z is

$$\mathbf{f}_{\perp} = \mathbf{f}^a \ell_a = \bar{\kappa} \left[\frac{K_0}{a \cosh^2(z/a)} - \frac{1}{2}K_0^2 \right] \mathbf{l}. \quad (72)$$

From this we note that if spontaneous curvature K_0 vanishes, there is no stress induced by bending. Because of its sign, the term $-\bar{\kappa}K_0^2/2$ has the effect of decreasing the area in the same way as the surface tension does. Moreover, the effect of the first term on the right-hand side of equation (72) depends on the sign of K_0 [26].

Once again, the radial forces are canceled out when integrating around the loop. Hence, the required force to build a nematic catenoid up to a loop of radius $r < R$ and by considering (68) is

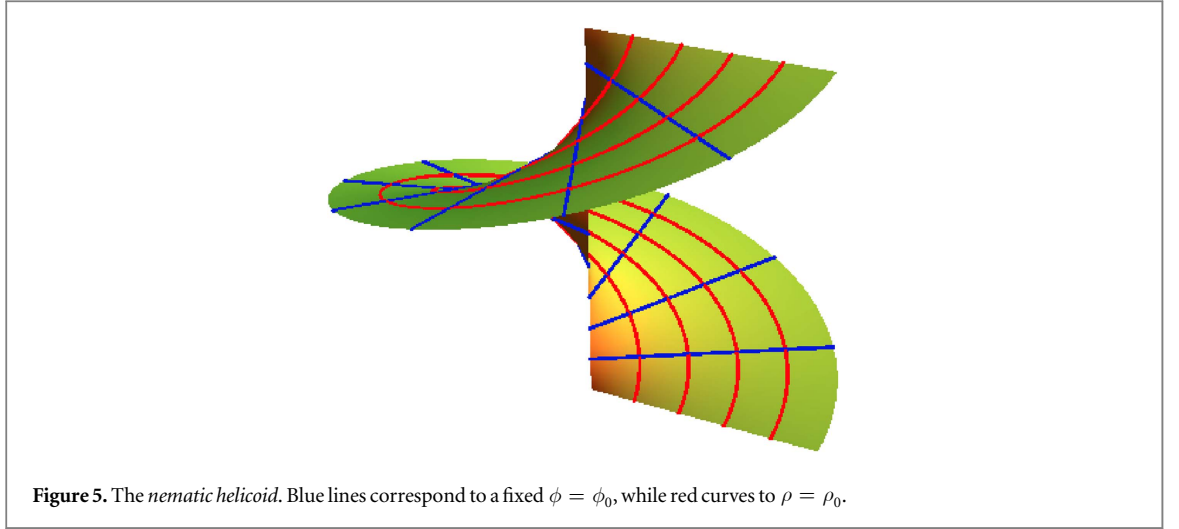


Figure 5. The nematic helicoid. Blue lines correspond to a fixed $\phi = \phi_0$, while red curves to $\rho = \rho_0$.

$$\mathbf{F} = 2\pi a \left[\sigma - \frac{2\kappa}{a^2} - \bar{\kappa} \left(\frac{K_0}{a} - \frac{K_0^2}{2} \right) \right] \mathbf{k}, \quad (73)$$

where $\mathbf{k} = (0, 0, 1)$. Therefore the induced ordered tension, acting on horizontal loops $\sigma_r = 2\kappa/a^2$, could be relevant in the spontaneous formation of tubular membranes, as this adds to the *spontaneous tension* $\sigma_s = \kappa K_0(1/a - K_0/2)$ [26].

Is worth mentioning the case of the helicoid as it is a surface related to catenoid by an isometry (see figure 5). Let us see briefly some details about this case. The helicoid is a minimal surface that can be parametrized by

$$\begin{aligned} \mathbf{X}(\rho, \phi) &= (\rho \cos \phi, \rho \sin \phi, p \phi), \\ &= \rho \boldsymbol{\rho} + p \phi \mathbf{k}, \end{aligned} \quad (74)$$

where $\boldsymbol{\rho} = (\cos \phi, \sin \phi, 0)$ is the unit vector along the radial direction. We take $0 \leq \rho \leq R$; $0 \leq \phi \leq 2\pi n$ and $p > 0$. The tangent vectors to the helicoid are $\mathbf{e}_\rho = \boldsymbol{\rho}$ and $\mathbf{e}_\phi = \rho \boldsymbol{\phi} + p \mathbf{k}$, where $\boldsymbol{\phi} = (-\sin \phi, \cos \phi, 0)$ and therefore the induced metric can be written as

$$g_{ab} d\xi^a d\xi^b = d\rho^2 + (\rho^2 + p^2) d\phi^2. \quad (75)$$

The area element is $dA = \sqrt{\rho^2 + p^2} d\rho d\phi$. The normal unit vector to the surface is then $\mathbf{n} = \frac{1}{\sqrt{\rho^2 + p^2}} (p \boldsymbol{\phi} - \rho \mathbf{k})$ and the scalar curvature

$$\mathcal{R} = -\frac{2p^2}{(\rho^2 + p^2)^2}. \quad (76)$$

The induced force onto the surface is given by (46) where the components of the stress are $\mathbf{f}^\rho = -(\sigma + \kappa \mathcal{R}) \boldsymbol{\rho}$ and $\mathbf{f}^\phi = -\frac{(\sigma + \kappa \mathcal{R})}{\rho^2 + p^2} (\rho \boldsymbol{\phi} + p \mathbf{k})$. The force on *meridians* (red curves in figure 5) is $\mathbf{F} = \int d\phi \sqrt{g} \mathbf{f}^\rho$, that vanishes when integrated over a period interval. The force on *rulings* (blue lines in figure 5) is given by $\mathbf{F} = \int_0^R d\rho \sqrt{g} \mathbf{f}^\phi$; the curvature is added to the surface tension to decrease the area of the helicoid.

6. Perturbations

We can now obtain some results about the stability operator of nematic membranes. Those are obtained from the second variation of the energy. By taking a variation in equation (34) we have [27]

$$\begin{aligned} \delta_\perp^2 F &= \int d^2\xi \delta [\sqrt{g} K (\sigma + \kappa \mathcal{R})] \Phi, \\ &= \int dA \Phi \mathcal{L} \Phi, \end{aligned} \quad (77)$$

where the stability operator is given by

$$\mathcal{L} = -[\sigma + \kappa(\mathcal{R} + 2K^2)] \nabla_g^2 + 2\kappa K K^{ab} \nabla_a \nabla_b + [\sigma + \kappa(\mathcal{R} - K^2)] \mathcal{R}. \quad (78)$$

It can be written in an explicit self-adjoint way as

$$\mathcal{L} = \nabla_a A^{ab} \nabla_b + B, \quad (79)$$

Table 1. The eigenvalues ϵ for some values of $m = 0, 1$ and $n = 0, 1, 2, 3, 4, 5$.

mn	0	1	2	3	4	5
0	-2	14/9	46/25	94/49	158/81	238/121
1	1	17/9	49/25	97/49	161/81	241/121

where

$$\begin{aligned} A^{ab} &= 2\kappa K (K^{ab} - g^{ab}K) - g^{ab}(\sigma + \kappa\mathcal{R}), \\ B &= [\sigma + \kappa(\mathcal{R} - K^2)]\mathcal{R} + 2\kappa(K^{ab} - g^{ab}K)\nabla_a\nabla_b K - \kappa\nabla_g^2\mathcal{R}. \end{aligned} \quad (80)$$

Eigenvalues of this self-adjoint operator are real numbers. We can also check that a normal translation is a zero mode of \mathcal{L} . To see this take $\delta\mathbf{X} = \mathbf{a}$, a normal translation such that $\Phi = \mathbf{a} \cdot \mathbf{n}$ and $\mathbf{a} \cdot \mathbf{e}_a = 0$. Then we can write $\nabla_a\Phi = \mathbf{a} \cdot K^b{}_a\mathbf{e}_b$. Taking a further derivative we see that $\nabla_a\nabla_b\Phi = -K^c{}_b K_{ab} \mathbf{a} \cdot \mathbf{n}$. We also have that

$$\nabla_g^2\Phi = -K^{ab}K_{ab}\Phi. \quad (81)$$

Hence, substituting this results into equation (78) we find that $\mathcal{L}\Phi = K\mathcal{E}$, and therefore on equilibrium shapes it is a zero mode. On minimal surfaces the stability operator takes the form

$$\mathcal{L}_m = (\sigma + \kappa\mathcal{R})(-\nabla_g^2 + \mathcal{R}). \quad (82)$$

Solving the eigenvalue equation $\mathcal{L}_m f = Ef$ helps us to analyze the stability of the surface. Let us look at the example of the catenoid given in previous section, that is a minimal surface of revolution with negative curvature equation (66), figure 4. In this example, taking advantage of the azimuthal symmetry, we express the solution as $f(z, \phi) = e^{im\phi}Z(z)$. Moreover, using the fact that $\phi(0) = \phi(2\pi)$, we get that $|m| = 0, 1, 2 \dots$. If we rename $u = z/a$, we can write the $E = 0$ equation:

$$Z''(u) + \frac{2}{\cosh^2 u}Z(u) = m^2Z(u). \quad (83)$$

The solution of this equation with $m = 0$ is, $Z(u) = C(u \tanh u - 1) + C_0 \tanh u$. The non-trivial critical point such that $Z(u_c) = 0$, appears if we take $C_0 = 0$; one can find that $u_c \sim \pm 1.19$ [29]. If we select the positive value then the conditions for the catenoid to exist, are given by $u < u_c$ and $\rho < \rho_c$, which in turns implies that $u/\rho < u_c/\rho_c$ and therefore we obtain the well known result, $h/R < 0.66$. In this way, $z < a$ is a condition that must be fulfilled for all $z < h$.

In order to find solutions if $E \neq 0$, we expand up to quadratic order in u . With this we are able to rewrite the eigenvalue equation as

$$-Z''(u) + m^2Z - 2(1 - u^2)Z = \epsilon u^2Z, \quad (84)$$

where the dimensionless quantity $\epsilon = Ea^4/(\sigma - 2\kappa/a^2)$ has been defined. This equation is known as the parabolic cylinder equation and can be written in its standard way if we change the variable $\mu = \sqrt{2}(2 - \epsilon)^{1/4}u$, as well as identify $\nu = (2 - m^2 - \sqrt{2 - \epsilon})/(2\sqrt{2 - \epsilon})$. Note that since ν is a real number, imposes the upper bound $\epsilon < 2$. The most general solution to equation (84), is therefore given by $Z(\mu) = C_1 D_\nu(\mu) + C_2 D_{-\nu-1}(-i\mu)$, where C_1 and C_2 are constants and the parabolic cylinder function $D_\nu(\mu)$ appears, [30]. We take the boundary condition such that $Z \rightarrow 0$ as $|\mu| \rightarrow \infty$. The function $D_{-\nu-1}(-i\mu)$ grows exponentially if $u \rightarrow \infty$, so we choose $C_2 = 0$.

On the other hand if $u \rightarrow -\infty$, the behavior of the function is $D_\nu(u) \sim -\frac{u^{-\nu-1}e^{u^2/4}}{\Gamma(-\nu)}[1 + (\nu + 1)(\nu + 2)/2z^2 + \dots]$, thus in this limit this function diverges unless $\nu = 0, 1, 2 \dots$. A similar setting appears when we solve the Schrödinger equation for the harmonic oscillator. Accordingly the fact that ν takes non-negative integer values implies that

$$\epsilon = 2 - \left(\frac{2 - m^2}{1 + 2n}\right)^2, \quad (85)$$

where $n = 0, 1, 2, \dots$, and $|m| = 0, 1$. In table 1 we show some of the eigenvalues ϵ and in figure 6 some of the eigenfunctions. We see that eigenmodes with $m = 0$ includes the negative value $\epsilon = -2$, that is, the *nematic catenoid* is unstable under these symmetric perturbations. On the other hand, the stability of the catenoid under $m = 1$ modes is evident. Of course this happens in the surface tension regime since, as was discussed by Giomi [17], in the orientational regime, to be analyzed, the surface should have a constant negative curvature.

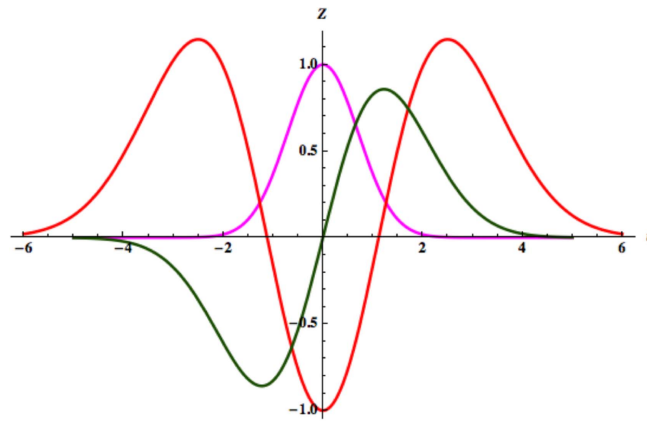


Figure 6. The symmetric $Z(\mu)$ functions (i.e. $m = 0$) for $n = 0, 1, 2$. The greater the value of n , the greater the number of nodes. The *nematic catenoid* is unstable under these perturbations.

7. Concluding remarks

Taking into account in-plane ordering configuration of a vector field on a curved surface we have found the corresponding shape equations $K(\sigma + \kappa\mathcal{R}) = 0$ of the surface. We find that in the regime where the bending energy is negligible not only minimal surfaces remains as solutions, but hyperbolic-like surfaces also appear among others such that developable surfaces. It should be noted that this is possible even without the presence of topological defects. Starting from the invariance of the energy under Euclidean transformations, we have found the stress as $\mathbf{f}^a(\xi) = -g^{ab}(\sigma + \kappa\mathcal{R}(\xi))\mathbf{e}_b$. The fact that this vector field is tangent to the surface is not surprising since it comes from a purely intrinsic energy. Because of this, the induced torque is only external. The stress is a vector field that vanishes at each point in a hyperbolic-like membranes, $\mathcal{R} = -\sigma/\kappa$. In the presence of topological defects with charge q_i , the stress is modified as $\mathbf{f}^a(\xi) = -g^{ab}[\sigma + \kappa\rho(\xi)]\mathbf{e}_b$, where $\rho(\xi) = \mathcal{R} - \sum_i q_i \delta(\xi, \xi_i)$.

In the case of the *nematic catenoid* the force takes its largest value at the waist of the surface and decreases as we move. Furthermore, we study the stability of the catenoid under azimuthal asymmetric perturbations, by showing the positiveness of the eigenvalues of the stability operator. In contrast, the catenoid is unstable under azimuthal symmetric perturbations. Several other geometries of interest can be studied. In the case of a closed surface one can take into account the volume inside through a Lagrange multiplier P . In this case one have to take into account that, due to the Poincaré–Hopf theorem, the total topological charge on the surface has the same value as the Euler invariant $\chi = 2$. Thereby, there are topological defects to take into account, [31]. We return to this issue elsewhere.

Acknowledgments

JAS want to thank J Guven the many fruitful discussions on geometry along the way. GB acknowledges CONACyT Mexican Council of Science and Technology for a scholar fellowship No. 372378. This work was partially supported by DSA under Grant No. UAM-C-CA-38, IDCA 23886.

ORCID iDs

O González-Gaxiola  <https://orcid.org/0000-0003-3317-9820>

J A Santiago  <https://orcid.org/0000-0003-4553-8253>

References

- [1] Lubensky T C and Prost J 1992 *J. Phys.* II **2** 371
- [2] Serra F 2016 *Liq. Cryst.* **43** 1920–36
- [3] Leo van Hemmen J and Leibold C 2007 *Phys. Rep.* **444** 51
- [4] Seifert U 1997 *Adv. Phys.* **46**:1 13–7
- [5] Turner A M, Vitelli V and Nelson D D 2010 *Rev. Mod. Phys.* **82** 1301
- [6] David F 1989 *Statistical Mechanics of Membranes and Surfaces* ed D Nelson et al (Singapore: World Scientific)
- [7] Vitelli V and Nelson D R 2004 *Phys. Rev. E* **70** 051105
- [8] Xing X, Shin H, Bowick M A, Yao Z, Jia L and Li M 2012 *Proc. Natl Acad. Soc.* **109** 5202

- [9] Nelson D and Peliti L 1987 *J. Phys.* **48** 1085
- [10] Napoli G and Vergori L 2010 *J. Phys. A: Math. Theor.* **43** 445207
- [11] Napoli G and Vergori L 2012 *Phys. Rev. E* **85** 061701
- [12] Lenz P and Nelson D 2003 *Phys. Rev. E* **67** 031502
- [13] Kamien R D, Nelson D, Santangelo C and Vitelli V 2009 *Phys. Rev. E* **80** 051703
- [14] Turner A M, Vitelli V and Nelson D 2010 *Rev. Mod. Phys.* **82** 1301
- [15] Niv I and Efrati E 2017 Geometric frustration and compatibility conditions for two dimensional director fields arXiv:1706.08187
- [16] Machon T and Alexander G P 2014 *Phys. Rev. Lett.* **113** 027801
- [17] Giomi L 2012 *Phys. Rev. Lett.* **109** 136101
- [18] Frank J R and Kardar M M 2008 *Phys. Rev. E* **77** 041705
- [19] Jiang H, Huber G, Pelcovits R and Powers T 2007 *Phys. Rev. E* **76** 031908
- [20] Capovilla R and Guven J 2002 *J. Phys. A: Math. Gen.* **35** 6233
- [21] do Carmo M 1976 *Differential Geometry of Curves and Surfaces* (Englewood Cliffs, NJ: Prentice-Hall)
- [22] Napoli G and Vergori L 2012 *Phys. Rev. Lett.* **108** 207803
- [23] Kamien R D 2002 *Rev. Mod. Phys.* **74** 953
- [24] Polyakov A M 1981 *Phys. Lett. B* **103** 207
- [25] Guven J 2004 *J. Phys. A: Math. Gen.* **37** L313
- [26] Lipowsky R 2013 *Faraday Discuss.* **161** 305331
- [27] Capovilla R, Guven J and Santiago J A 2003 *J. Phys. A: Math. Gen.* **36** 6281
- [28] Mbanga B D, Grason G M and Santangelo C D 2012 *Phys. Rev. Lett.* **108** 017801
- [29] Durand L 1981 *Am. J. Phys.* **49** 334–43
- [30] Abramowitz M and Stegun I 1964 *Handbook of Mathematical Functions* (New York: Dover)
- [31] Vitelli V and Turner A M 2004 *Phys. Rev. Lett.* **93** 215301